UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 4 June, 2004 1.30 to 4.30

PAPER 59

ADVANCED COSMOLOGY

Attempt **THREE** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In the 3+1 formalism for General Relativity, one selects a set of spacelike surfaces Σ_3 which foliate spacetime, with a timelike normal n^μ normalised so that $n^\mu n^\nu g_{\mu\nu} = -1$. The projection operator onto the tangent space of Σ_3 is $P^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu$, and the extrinsic curvature of Σ_3 is $K_{\alpha\beta} = P^\mu_\alpha \nabla_\mu n_\beta \in \Sigma_3$, where ∇_μ is the four dimensional covariant derivative. The three dimensional covariant derivative D_μ is given by applying ∇_μ and then projecting all tensor indices into Σ_3 using P^μ_ν.

(a) Show that $P^\mu_\alpha P^\alpha_\delta = P^\mu_\delta$, and $P^\mu_\alpha n_\mu = 0$.

(b) Show that $P^\alpha_\mu P^\beta_\nu \nabla_\alpha P^\epsilon_\beta = K_{\mu\nu} n^\epsilon$.

(c) From the identity

$$D_\mu D_\nu W_\gamma - D_\nu D_\mu W_\gamma = W_\lambda {}^{(3)}R^\lambda_{\gamma\nu\mu}, \qquad (*)$$

for any $W_\gamma \in \Sigma_3$ i.e. $W_\gamma n^\gamma = 0$, show that

$$^{(3)}R^\lambda_{\gamma\nu\mu} = P^\lambda_\xi P^\alpha_\mu P^\beta_\nu P^{\delta}_\gamma {}^{(4)}R^\xi_{\delta\beta\alpha} - K_{\mu\gamma} K^\lambda_\nu + K_{\nu\gamma} K^\lambda_\mu \, .$$

You may assume $K_{\mu\nu}$ is symmetric. [*Hint*: express the left hand side of (*) in four dimensional terms.]

2 Consider a massless scalar field ϕ in a contracting, flat FRW universe.

(a) Assuming the background field $\phi_0(\tau)$ is spatially homogeneous, show that

$$a(\tau) = (-\tau)^{1/2}, \qquad \phi_0(\tau) = \sqrt{3/2}\ln(-\tau), \qquad -\infty < \tau < 0\,,$$

where τ is the conformal time, solves the Friedmann equation and the scalar field equation in units where $8\pi G = 1$.

(b) In conformal Newtonian gauge, the perturbed metric is

$$ds^2 = a^2(\tau)[-(1+2\Phi)d\tau^2 + (1-2\Psi)d\vec{x}^2]$$

(for scalar perturbations). The scalar field is $\phi = \phi_0(\tau) + \delta\phi$. Show that the perturbed stress-energy tensor has components

$$a^2 T^0_0 = -\frac{1}{2}\phi_0'^2 - \phi_0'\delta\phi' + \phi_0'^2\Phi$$
$$a^2 T^0_i = -[\phi_0'\delta\phi_{,i}]\,,$$

to linear order in the perturbations, where primes denote derivatives with respect to τ.

(c) The perturbed Einstein equations read

$$2[-3\mathcal{H}(\mathcal{H}\Phi + \Psi') + \nabla^2\Psi] = -\phi_0'^2\Phi + \phi_0'\delta\phi' \qquad (1)$$
$$2(\mathcal{H}\Phi + \Psi') = \phi_0'\delta\phi \qquad (2)$$

where $\mathcal{H} = a'/a$ and $\Phi = \Psi$ since there is no anisotropic stress to linear order.

Show that these equations and the background equations imply that

$$\Phi'' + 6\frac{a'}{a}\Phi' - \nabla^2\Phi = 0 \qquad (3)$$

(d) Writing $\Phi = \sum_{\mathbf{k}} \Phi_{\mathbf{k}}(\tau)e^{i\mathbf{k}\cdot\mathbf{x}}$ and similarly for $\delta\phi$, show first that (3) has the positive frequency solution

$$\Phi_{\mathbf{k}}(\tau) \sim \frac{1}{a^3}e^{-ik\tau}, \qquad k\tau \to -\infty\,,$$

and that (1) and (2) are consistent with this as long as

$$\delta\phi_{\mathbf{k}}(\tau) \sim \frac{1}{a}e^{-ik\tau}, \qquad k\tau \to -\infty\,.$$

(e) If $\delta\phi_k \sim \frac{1}{a}\sqrt{\frac{\hbar}{2k}}e^{-ik\tau}$ is the correctly normalised positive frequency incoming mode as $k\tau \to -\infty$, calculate the spectral index of $\langle|\Phi_k|^2\rangle$ (i.e. the power of k) as $\tau \to 0$.

3 In a flat FRW universe ($\Omega = 1$), in synchronous gauge (specifying metric perturbations with $h^{0\mu} = 0$), the perturbations of a multicomponent fluid obey the following evolution equations

$$\begin{aligned} &\delta_N' + (1+w_N)i\mathbf{k}\cdot\mathbf{v}_N + \tfrac{1}{2}(1+w_N)h' = 0\,, \\ &\mathbf{v}_N' + (1-3w_N)\frac{a'}{a}\mathbf{v}_N + \frac{w_N}{1+w_N}i\mathbf{k}\delta_N = 0\,, \\ &h'' + \frac{a'}{a}h' + 3\left(\frac{a'}{a}\right)^2\sum_N(1+3w_N)\Omega_N\delta_N = 0\,, \end{aligned} \qquad (\dagger)$$

where δ_N is the density perturbation, Ω_N is the fractional density, $\mathbf{v}_N$ is the velocity and $P_N = w_N\rho_N$ is the equation of state of the Nth fluid component, and $\mathbf{k}$ is the comoving wavevector ($k = |\mathbf{k}|$), h is the trace of the metric perturbation and primes denote differentiation with respect to conformal time τ ($d\tau = dt/a$).

(a) Suppose that the universe is composed of two components, (comoving) cold dark matter ρ_C with no pressure ($P_C = 0$) and a radiation fluid ρ_R with equation of state $P_R = \rho_R/3$. Show that the coupled matter-radiation equations arising from (†) become

$$\begin{aligned} &\delta_C'' + \frac{a'}{a}\delta_C' - \frac{3}{2}\left(\frac{a'}{a}\right)^2(\Omega_C\delta_C + 2\Omega_R\delta_R) = 0\,, \\ &\delta_R'' + \frac{1}{3}k^2\delta_R - \frac{4}{3}\delta_C'' = 0\,. \end{aligned}$$

Consider times well before equal matter-radiation (i.e. $\tau \ll \tau_{\rm eq}$ when $\rho_R = \rho_C$), to find approximate growing mode solutions for both matter and radiation density perturbations which are initially adiabatic ($\delta_R = \frac{4}{3}\delta_C$) in the limits $\tau \ll 2\pi/k$ and $\tau \gg 2\pi/k$.

(b) Now consider the late universe filled with cold dark matter and a small baryonic component with $P_B = c_{\rm s}^2\rho_B$ ($c_{\rm s} << 1$), that is, at times $\tau \gg \tau_{\rm eq}$ when $\Omega_C + \Omega_B \approx 1$ (ignoring radiation and Λ). By making appropriate approximations, show that the evolution equations (†) can be reduced to

$$\begin{aligned} &\delta_C'' + \frac{a'}{a}\delta_C' - \frac{3}{2}\left(\frac{a'}{a}\right)^2(\Omega_C\delta_C + \Omega_B\delta_B) = 0\,, \\ &\delta_B'' + \frac{a'}{a}\delta_B' - \left[\frac{3}{2}\left(\frac{a'}{a}\right)^2(\Omega_C\delta_C + \Omega_B\delta_B) - c_{\rm s}^2k^2\delta_B\right] = 0\,. \end{aligned}$$

For a small baryon density ($\Omega_B << \Omega_C$) and initially adiabatic perturbations, show that baryonic structures can only grow on physical wavelengths greater than,

$$\lambda_{\rm J} \approx c_{\rm s}\left(\frac{\pi}{G\bar{\rho}_C}\right)^{1/2}\,,$$

where $\bar{\rho}_C$ is the homogeneous cold dark matter density. Qualitatively describe the evolution of large-scale baryonic perturbations in the matter era ($\tau > \tau_{\rm eq}$), given that $c_{\rm s} \approx 1/\sqrt{3}$ prior to photon decoupling ($\tau < \tau_{\rm dec}$) and $c_{\rm s} \approx 10^{-5}(T/T_{\rm dec})$ afterwards ($\tau > \tau_{\rm dec}$).

4 In an expanding universe we can expand the photon distribution function $f(\mathbf{x},\mathbf{p},\tau)$ about the Planck spectrum $f_0(p,\tau) = f_0(q)$ as

$$f(\mathbf{x},\mathbf{p},\tau) = f_0(q) + f_1(\mathbf{x},q,\hat{\mathbf{n}},\tau)\,,$$

where the comoving momentum $q \equiv ap$ with $p = |\mathbf{p}|$ and the photon propagation direction is $\hat{\mathbf{n}} = \mathbf{p}/p$. The collisionless Boltzmann equation $df/d\lambda = 0$ then can be expressed as

$$\frac{\partial f_1}{\partial \tau} + \hat{n}^i \frac{\partial f_1}{\partial x^i} + \frac{dq}{d\tau}\frac{df_0}{dq} + \frac{dq}{d\tau}\frac{\partial f_1}{\partial q} + \frac{dn^i}{d\tau}\frac{\partial f_1}{\partial \hat{n}^i} = 0\,. \qquad (*)$$

(a) Consider photon propagation in synchronous gauge,

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (\delta_{ij} + h_{ij})dx^i dx^j\right]\,,$$

to show that the photon comoving momentum q must obey

$$\frac{dq}{d\tau} = -\frac{1}{2} q h'_{ij}\hat{n}^i\hat{n}^j\,,$$

where primes denote derivatives with respect to τ. [*Hint:* Use the geodesic equation $\frac{dp^0}{d\lambda} + \Gamma^0_{\nu\sigma}p^\nu p^\sigma = 0$ for which you may assume that $\Gamma^0_{00} = \frac{a'}{a}$, $\Gamma^0_{0i} = 0$ and $\Gamma^0_{ij} = \frac{a'}{a}(\delta_{ij} + h_{ij}) + \frac{1}{2}h'_{ij}$.]

Use this result to show that $(*)$ in Fourier space can be re-expressed as

$$\frac{\partial f_1}{\partial \tau} + ik\mu f_1 = \frac{1}{2} q \frac{df_0}{dq} h'_{ij}\hat{n}^i\hat{n}^j\,, \qquad (\ddagger)$$

where $\mu \equiv \hat{\mathbf{k}}\cdot\hat{\mathbf{n}}$ with the wavevector direction $\hat{\mathbf{k}} = \mathbf{k}/k$ and magnitude $k = |\mathbf{k}|$. Briefly explain why the last two terms in $(*)$ have been neglected.

(b) Consider the ansatz in which we express perturbations in the distribution function $f(\mathbf{x},\mathbf{p},\tau)$ by deviations of the temperature $T(\mathbf{k},\hat{\mathbf{n}},\tau)$ about the mean temperature $\bar{T}(\tau) \propto a^{-1}$ in the Planck spectrum. Show that

$$f(\mathbf{x},\mathbf{p},\tau) \equiv f_0\left(\frac{\bar{T}(\tau)q}{T(\mathbf{k},\hat{\mathbf{n}},\tau)}\right) \approx f_0(q) - q\frac{df_0}{dq}\frac{\Delta T}{T}(\mathbf{k},\hat{\mathbf{n}},\tau)\,.$$

Defining the brightness function by $\Delta(\mathbf{k},\hat{\mathbf{n}},\tau) = 4\frac{\Delta T}{T}$, integrate out the q-dependence in $(\ddagger)$ to show that

$$\Delta' + ik\mu\Delta = -2h'_{ij}\hat{n}^i\hat{n}^j\,. \qquad (**)$$

It is usual to solve for Δ in $(**)$ using a moment expansion in the parameter μ; however, prior to decoupling while the photons are very close to equilibrium, explain why we need only consider the first two terms in that expansion.

(c) The solution today at τ_0 of $(**)$ can be expressed in terms of the initial conditions at photon decoupling $\tau_{\rm dec}$ and a line integral over the metric perturbations (*you need not derive this result*):

$$\frac{\Delta T}{T}(\mathbf{x},\hat{\mathbf{n}},\tau_0) = \frac{1}{4}\delta_\gamma(\mathbf{x},\tau_{\rm dec}) + \hat{\mathbf{n}}\cdot\mathbf{v}_\gamma(\mathbf{x},\tau_{\rm dec}) - \frac{1}{2}\int_{\tau_{\rm dec}}^{\tau_0} h'_{ij}\hat{n}^i\hat{n}^j d\tau\,.$$

Briefly discuss the physical origin of each of these three terms and the angular scales on which they are typically important. Sketch a diagram illustrating their contributions to the angular power spectrum.

5 Consider the spacetime

$$ds^2 = -\cosh^2 t \; dt^2 + dx^2 .$$

(a) Show that the timelike geodesics emanating from $t = 0$, $x = 0$ are periodic and describe their behaviour physically.

[*Hint*: You may find the following integral useful:

$$\int_0^{\pi} \frac{dz}{1 + A \sin^2 z} = \frac{\pi}{\sqrt{1+A}} .$$

]

(b) By means of an appropriate conformal transformation, describe the causal structure of this spacetime.

(c) Is the spacetime globally hyperbolic? Justify your answer and describe its physical consequences.

6 (a) Consider a static hyperbolic spacetime described by the metric

$$ds^2 = -dt^2 + d\xi^2 + R^2 \sinh^2[\xi/R](d\theta^2 + \sin\theta d\phi^2) ,$$

where the length R characterizes the scale of the spatial curvature of the constant time slices.

We shall consider how classical electrodynamics in this spacetime is altered by the presence of spatial curvature. You may assume that Gauss's law remains valid. Compute the electric field of a point charge as a function of distance R.

(b) Consider a massless scalar field $\phi(\mathbf{x})$ in the Kaluza-Klein spacetime

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + dw^2$$

where $w = 0$ and $w = R$ are identified by means of periodic boundary conditions.

Express the retarded propagator for the field $\phi(\mathbf{x})$, $G(\mathbf{x}, \mathbf{x}')$ satisfying

$$\Box G(\mathbf{x}, \mathbf{x}') = (\partial_t^2 - \nabla^2 - \partial_w^2) G(\mathbf{x}, \mathbf{x}') = \delta^5(\mathbf{x} - \mathbf{x}')$$

as a sum and/or integral in momentum space, where

$$\delta^5(\mathbf{x} - \mathbf{x}) = \delta(x - x')\delta(y - y')\delta(z - z')\delta(w - w')\delta(t - t') .$$

(c) Explain how $G(\mathbf{x}, \mathbf{x}')$ at $w = w'$ may be interpreted as an infinite tower of species of Kaluza-Klein excitations of differing mass.

(d) Discuss the limits where $|\mathbf{x} - \mathbf{x}'|$, $|t - t'| \ll R$ and $|\mathbf{x} - \mathbf{x}'|, |t - t'| \gg R$ when $w = w'$ as well as the physical interpretation of these two limiting cases.