UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 3 June, 2004 9 to 12

PAPER 58

APPLICATIONS OF DIFFERENTIAL GEOMETRY TO PHYSICS

Attempt **FOUR** *questions.*

There are **seven** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the Killing form B on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ of a Lie group G. Give a condition on G that the Killing form B be non-degenerate (i.e. invertible). Explain how, in this case, one may endow G with a bi-invariant pseudo-riemannian metric and show that this metric is an Einstein metric.

Comment briefly, giving illustrations but no proofs, on the signature of the Killing form B and its relation to the topology of the group G.

2 Define the terms *orbit*, *stabilizer*, *transitive*, *simply transitive* and *multiply transitive* for the action of a group G acting on a manifold X. Show that if the action is multiply transitive, then $X = G/H$ for some subgroup $H \subset G$. How is the subgroup H determined?

Show that the space of quantum states of an $(n+1)$-state quantum system is complex projective space $\mathbb{CP}^n$. Show that $\mathbb{CP}^n \equiv U(n+1)/U(n) \times U(1)$.

Give a description of S^{2n+1} as an S^1 bundle over $\mathbb{CP}^n$.

3 Define a Poisson manifold $\{P, \omega^{\mu\nu}\}$ and derive the condition that the second rank antisymmetric contravariant tensor field $\omega^{\mu\nu}$ must satisfy in order that the associated Poisson bracket satisfies the Jacobi identity.

Define a symplectic manifold, and show that every symplectic manifold is a Poisson manifold.

Show, by means of an example based on the dual $\mathfrak{g}^\star$ of a Lie algebra $\mathfrak{g}$, that not every Poisson manifold, equipped with a Poisson bracket satisfying the Jacobi identity, need be a symplectic manifold.

4 The metric of three-dimensional Anti-de-Sitter spacetime AdS_3 may be written in globally static coordinates as

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{1+r^2} + r^2 d\theta^2,$$

with both t and θ being periodic coordinates of period 2π. By means of the embedding into $\mathbb{E}^{2,2}$ given by

$$X^0 = \sqrt{1+r^2}\sin t\,, \qquad X^4 = \sqrt{1+r^2}\cos t\,,$$
$$X^1 = r\cos\theta\,, \qquad X^1 = r\sin\theta\,,$$

show that the metric coincides, up to a constant factor, with the Killing metric on $SL(2,\mathbb{R})$.

Show how $SO(2,2)$ acts by isometries and construct a $2:1$ homomorphism from $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ to $SO(2,2)$.

What can you say about the group $SO(2,1)$?

5 Define a principal fibre bundle and show that it admits a global right action of the structural group G. Show also that such a bundle admits a global section if and only if it is trivial.

Illustrate your answer by means of the bundle of pseudo-orthonormal frames of a pseudo-riemannian manifold.

In particular, show that $SO(4,1)$ is the bundle of orthonormal frames for four-dimensional De-Sitter spacetime dS_4.

Given that S^3 admits a global frame field, does the bundle of pseudo-orthonormal frames for dS_4 admit a global section?

6 Let $\{M,g\}$ be a pseudo-riemannian manifold with a local pseudo-orthonormal basis for the tangent space $\mathbf{e}_a$ and dual basis ω^a. Establish the equations

$$\theta_{ab} + \theta_{ba} = 0,$$
$$d\omega^a + \theta^a{}_b \wedge \omega^b = 0\,,$$
$$R^a{}_b = d\theta^a{}_b + \theta^a{}_c \wedge \theta^c{}_b\,,$$

where $\nabla \mathbf{e}_a = \mathbf{e}_b \otimes \theta^b{}_a$, and $R^a{}_b$ is the curvature 2-form of the Levi-Civita connection.

How does $R^a{}_b$ change under change of basis?

7 Write a brief essay on integration on manifolds, giving applications to topological conservation laws and the gauge-invariant coupling of a p-brane to a $(p+1)$-form potential.