UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 4 June, 2004 9 to 12

PAPER 57

BLACK HOLES

Attempt **THREE** *questions.*

The are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write an essay describing the laws of black hole mechanics and their relationship to the corresponding laws of thermodynamics.

2 Describe the concept of a vierbein in general relativity.

Describe how to use the vierbein to construct curved spacetime gamma-matrices given a set of flat space gamma-matrices γ^μ.

Show that

$$\{\gamma_a, \gamma_b\} = 2g_{ab}\ \mathbf{1},$$

where g_{ab} is the metric tensor, and $\mathbf{1}$ is the unit 4×4 matrix.

Show that

$$R_{abcd}\gamma^a\gamma^b\gamma^c\gamma^d = -2R\ \mathbf{1},$$

where R_{abcd} is the Riemann tensor.

Suppose that a spinor field ψ obeys the Dirac equation $\gamma^a \nabla_a \psi = 0$. Show that ψ obeys the wave equation

$$(\Box - \frac{1}{4}R)\psi = 0.$$

3 A three-dimensional spacelike surface Σ, given by $f(x) = 0$, is embedded in a four-dimensional spacetime with metric tensor g_{ab}. Describe how to construct the second fundamental form K_{ab} of this surface.

Derive the Gauss equation for the curvature tensor ${}^{(3)}R_{abcd}$ of Σ in terms of the curvature tensor ${}^{(4)}R_{abcd}$, the metric g_{ab}, and the second fundamental form K_{ab}.

Find an expression for the Ricci scalar ${}^{(3)}R$.

Suppose that the metric g_{ab} obeys the vacuum Einstein equations, and the surface Σ is totally umbilic. Show that ${}^{(3)}R \geq 0$.

[*Hint*: A totally umbilic surface is one where the second fundamental form is proportional to the induced metric on the surface.]

4 The metric of a spherically symmetric isolated charged gravitating object in five spacetime dimensions is, in Schwarzschild-like coordinates,

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2),$$

where (θ, ϕ, ψ) are coordinates on a three-sphere, and

$$V(r) = 1 - 2M/r^2 + Q^2/r^4$$

where $M > 0$ is the mass of the object and Q is the charge.

i) What conditions on Q and M are necessary for this metric to admit horizons?

ii) If the metric admits horizons, can these horizons ever be degenerate?

iii) Sketch the Penrose diagram(s) for the spacetime(s) described by this metric.

iv) Consider the (t, r)-plane. By putting $t = i\tau$, and looking at the geometry of the (τ, r) plane, describe the nature of the singularity encountered at r_0 where $V(r_0) = 0$ provided that $V'(r_0) \neq 0$.

v) What conditions on τ need to be imposed to remove this singularity?

vi) Comment on the relationship of your findings to Hawking's result that black holes have a non-zero temperature.