UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 31 May, 2004 9 to 12

PAPER 55

GENERAL RELATIVITY

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

Candidates may make free use of the information on the attached sheet.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Explain what is meant by a *linear connection* and a *covariant derivative* on a manifold. The connection is symmetric. What does this mean?

The *commutator* of the covariant derivative is defined by

$$\Delta_{ab} = [\nabla_a, \nabla_b] = 2\nabla_{[a}\nabla_{b]}\,.$$

If Q_1, Q_2 are tensor fields of the same type and R is a tensor field (all indices have been suppressed) show that

$$\Delta_{ab}(Q_1 + Q_2) = \Delta_{ab}Q_1 + \Delta_{ab}Q_2\,,$$

$$\Delta_{ab}(Q_1 R) = (\Delta_{ab}Q_1)R + Q_1\Delta_{ab}R\,,$$

and $\Delta_{ab}f = 0$ for scalar fields f.

If U^d is a vector field show that $\Delta_{ab}U^d$ is linear in U. This means that there exists a tensor $R_{abc}{}^d$, the *Riemann curvature tensor*, such that

$$\Delta_{ab}U^d = R_{abc}{}^d U^c\,.$$

Using this definition obtain similar formulae for $\Delta_{ab}\omega_c$ and $\Delta_{ab}S^c{}_{de}$ where ω_c is a covector field and $S^c{}_{de}$ is a $\binom{1}{2}$ tensor field. Show also that $R_{abc}{}^d = R_{[ab]c}{}^d$.

By considering the third covariant derivative of a scalar field f, obtain the *first Bianchi identity*

$$R_{[abc]}{}^d = 0\,.$$

Using a similar approach obtain the *second Bianchi identity*

$$\nabla_{[a}R_{bc]d}{}^e = 0\,.$$

2 State and discuss the *strong principle of equivalence.*

Give a heuristic derivation of the Einstein field equations

$$G_{ik} + \Lambda g_{ik} = -8\pi G\,T_{ik}\,.$$

[*You may use the result that the only non-vanishing component of the Newtonian Ricci tensor is* $R_{N00} = -\nabla^2\varphi$, *where* φ *is the gravitational potential.*]

3 A large thin spherical shell of mass M, radius R, rotates slowly about the z-axis with angular speed Ω. Here "large" means that $\epsilon = GM/R \ll 1$ and "slowly" means that $\Omega R = 0(\epsilon^{1/2})$. Introduce cartesian coordinates (t, x, y, z), a shell density $\rho = M\delta(r-R)/(4\pi R^2)$ where $r^2 = x^2+y^2+z^2$, and a 4-velocity $U^a = (1, -\Omega y, \Omega x, 0)$. The energy momentum tensor of the shell may be taken to be $T^{ab} = \rho U^a U^b$, and the line element inside the shell may be chosen to be

$$ds^2 = (1+2A)\,dt^2 - 2B_\alpha\,dt\,dx^\alpha - (1+2C)\gamma_{\alpha\beta}\,dx^\alpha\,dx^\beta,$$

to first order in ϵ, where $\gamma_{\alpha\beta}$ is the standard flat 3-metric.

Show that $-A = C = \epsilon$ and $B_\alpha = \omega(y, -x, 0)$ where $\omega = 4\epsilon\Omega/3$. Use your results to discuss timelike geodesics inside the shell and the *Lense-Thirring* effect.

[*You may use any information from the lecture handout included with this examination paper. You may use also the fact that, using spherical polar coordinates, each of* $\sin\theta\cos\phi$, $\sin\theta\sin\phi$ *and* $\cos\theta$ *is an eigenfunction of the flat space Laplacian* Δ *with eigenvalue* $-2/r^2$.]

4 Toy (1 + 1-dimensional) models with line elements for *de Sitter spacetime*

$$ds^2 = dt^2 - \cosh^2 t\,d\chi^2,$$

and *anti-de Sitter spacetime*

$$ds^2 = \cosh^2 r\,dt^2 - dr^2,$$

were introduced in the lectures. What are the ranges of the various coordinates?

Write an essay on the similarities and differences between these two models, paying particular attention to the geodesics, conformal structure and the horizons.