UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 3 June, 2004 1.30 to 4.30

PAPER 50

SUPERSYMMETRY AND EXTRA DIMENSIONS

Attempt **any three** *of questions 2,3,4,5,6 (weight* 20% *each)* **and attempt Question 1** *(weight* 40%*).*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Provide a short answer to each of the following questions in *no more* than 15 lines each:

(a) How do we label one-particle-state representations of the Poincaré group? How does that change with $N = 1$ supersymmetry? Provide the labels (only the labels) of the different states inside an arbitrary multiplet for both massive and massless states.

(b) Which problem is solved if supersymmetry is a symmetry of nature at energies close to $1TeV$? Why is extended supersymmetry not expected to be relevant at low energies?

(c) Explain the difference between the supersymmetric realization of the standard Higgs effect and the super-Higgs effect.

(d) Explain the similarities and differences between the standard Kaluza-Klein idea of extra dimensions and the Brane-World scenario. What is the maximum size of the extra dimensions in each of the scenarios?

2 Consider the Wess-Zumino model consisting of one chiral superfield Φ with tree-level Kähler potential $K_{tree} = \Phi^\dagger \Phi$ and tree-level superpotential

$$W_{tree} = \frac{1}{2} m \Phi^2 + \frac{1}{3} g \Phi^3 .$$

Taking the couplings g and m as spurion fields, define two $U(1)$ symmetries of the tree-level superpotential, with both $U(1)$'s acting on Φ, m, g and one of them acting also on the superspace coordinate θ (an R-symmetry).

Use these symmetries to infer the most general form that the (loop corrected) effective superpotential can take, in terms of an unspecified function $F(\Phi, m, g)$. Use the limits $g \to 0$, $m \to 0$ to determine the form of this arbitrary function and demonstrate that the superpotential is not renormalised.

What can we say about quantum corrections to the Kähler potential? Is holomorphy playing a role in this argument?

3 Use the $N=1$ supersymmetry algebra to prove that:

(a) In every supermultiplet, the number of bosons equals the number of fermions.

(b) The energy is non-negative and supersymmetry is broken if the energy of the vacuum is strictly positive.

Use the extended supersymmetry algebra to derive the BPS bound on massive states. Write down the properties that characterise BPS states.

[*Hint*: You may find useful to consider the anticommutator $\left\{ Q_\alpha^A - \Gamma_\alpha^A, \bar{Q}_{\dot\beta A} - \bar{\Gamma}_{\dot\beta A} \right\}$ with $\Gamma_\alpha^A \equiv \epsilon_{\alpha\beta} U^{AB} \bar{Q}_{\dot\gamma B} \left(\bar\sigma^0\right)^{\dot\gamma\beta}$ and U unitary.]

4 Demonstrate that the volume of a $N-1$ sphere of radius r is

$$V_{N-1} \;=\; \frac{2\pi^{N/2}}{\Gamma(N/2)}\, r^{N-1}.$$

[*Hint*: It may help to consider the integral $I_N = \int d^N x e^{-\rho^2}$ with $\rho^2 = \sum_{i=1}^N x_i^2$.]

Use this result to derive an expression for the electric (and gravitational) potential in D dimensions.

Show that the potential due to a point particle in five dimensions reduces to the 4-dimensional potential at distances much larger than the size of the fifth dimension.

5 Using the Dirac algebra in $D=2k$ dimensions, where k is a positive integer, find the dimensionality of the spinorial representation of the Lorentz group $SO(D-1,1)$. Is the representation irreducible? What is the difference for $D=2k+1$?

Dimensionally reduce the field content of D=11 supergravity to four-dimensions.

6 Consider $N=1$ supergravity with three chiral superfields S,T,C. The Kähler potential is in Planck units:

$$K \;=\; -\log(S+S^*) \;-\; 3\,\log\left(T+T^*-C^*C\right)$$

The superpotential is:

$$W \;=\; C^3 + a\, e^{-\alpha S} \;+\; b\,,$$

where a,b are arbitrary complex numbers and $\alpha>0$. Compute the scalar potential. Find the auxiliary field for S,T,C and verify that supersymmetry is broken. What is the value of the vacuum energy at its minimum? Are there flat directions? What is the gravitino mass?

[*Hint*: In supergravity the auxilary fields are proportional to the Kähler covariant derivatives $DW \;=\; \partial W/\partial\Phi + W\partial K/\partial\Phi$.]