UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 4 June, 2004 9 to 12

PAPER 5

MODULAR REPRESENTATIONS OF FINITE GROUPS

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

Throughout, G is a finite group and k a field of characteristic p dividing $|G|$.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 In this question, R will denote a coefficient ring (either k or the p-adic completion of a number ring). Define a p-block for RG and explain what it means for an indecomposable RG-module to lie in a block. Define also the defect group of a block B, and prove that every defect group D of B can be expressed as a Sylow intersection. Deduce that D contains every normal p-subgroup of G and that D is the largest normal p-subgroup of $N_G(D)$.

2 Let D be a p-subgroup of G. Define the Brauer map

$$\mathrm{Br}_D : (kG)^D \to kC_G(D)$$

showing that it is indeed a ring homomorphism and identify the kernel. State and prove Brauer's First Main Theorem (if you use any results in the proof you should state them clearly). If $DC_G(D) \leq H \leq G$ (but with no other restriction on H), and b a p-block of kH, use the First Main Theorem to define the Brauer correspondent b^G of b.

3 State Nagao's version of Brauer's Second Main Theorem. Use it to deduce that if B is a block with defect group D then there exists an indecomposable kG-module lying in B with vertex D and a trivial source.

Deduce further that if B is a block with defect group D, then B has finite representation type if and only if D is cyclic.

4 In this question, R will denote a coefficient ring (either k or the p-adic completion of a number ring). State and prove the Green Correspondence.

By a p-local RG-module we mean a direct sum of modules induced from subgroups of the form $N_G(D)$, where $D \neq 1$ is a p-group. Use the Green Correspondence to show that, if M is an indecomposable RG-module, there exist p-local RG-modules L_1, L_2, and projective RG-modules P_1, P_2 such that

$$M \oplus L_1 \oplus P_1 \cong L_2 \oplus P_2 \, .$$

5 Let B be a p-block of kG with defect group D, cyclic of order p^n ($n \geqslant 1$). Define the inertial index e of B. Assume that k is algebraically closed. Stating clearly any results you use, show that there are e simple modules in B and $p^n e$ indecomposable modules in B.

Explain what it means for a finite-dimensional k-algebra to be a Brauer tree algebra. Let Λ be a Brauer tree algebra. Let the simple Λ-modules $S_1, \ldots, S_r$ label the edges emanating from a vertex v and let them be in circular order as given. If v has multiplicity m (taking $m = 1$ if v is not the exceptional vertex) and $q \leq mr$ show that there is a uniserial Λ-module, unique up to isomorphism, of composition length q whose first composition factor is S_1, whose next one is S_2, and so on, using the circular ordering of the S_i.