UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 7 June, 2004 9 to 11

PAPER 49

STRING THEORY

Attempt **TWO** *questions.*

There are **three** *questions in total.*

The questions carry equal weight.

You will not be penalised for incorrect trivial numerical factors.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Derive the mass shell condition for the open string using:

$$L_0 \equiv \sum_{n=1}^{\infty} \alpha^{\mu}_{-n}\alpha_{n\mu} + \frac{1}{2}\alpha^{\mu}_{0}\alpha_{0\mu}$$

and the requirement that physical states satisfy:

$$(L_0 - 1)|state >= 0$$

The Virasoro algebra is:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}$$

What is the origin of the central charge c?

What are the conditions satisfied by physical states? Show that c is equal to the dimension of space-time.

Construct the level one open string states and discuss the implications of the physical state conditions. What is the interpretation of the physical and unphysical states?

[You may assume the following:

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty} :\alpha^{\mu}_{m-n}\alpha_{n\mu}: \quad , \quad [\alpha^{\mu}_{m}, \alpha^{\nu}_{n}] = m\delta_{m+n,0}\eta^{\mu\nu} \, .$$

]

2 Write down the action for a string coupled to an arbitrary background space-time metric, $g_{\mu\nu}$. The following metric is called a plane wave:

$$ds^2 = -2dx^+dx^- - m^2(\sum_I x_I x^I)(dx^+)^2 + \sum_I dx_I^2 \,,$$

where m is a constant mass parameter and $x^\pm = \frac{1}{\sqrt{2}}(x^0 \pm x^1)$ are light cone coordinates and $I = 2,..,25$.

Substitute this background space-time metric into the the closed string action with the choice for auxiliary world sheet metric being the flat Minkowski metric. Apply an additional fixing of the residual conformal diffeomorphism symmetry by choosing space-time x^+ to be world sheet time ie. $x^+ = \tau$. (This is essentially light cone gauge with $p^+ = 1$.)

Write out the resulting action and derive the equation of motion of the x^Is.

Solve the equation of motion to derive a mode expansion for the string in this background.

Write out the Hamiltonian and then substitute in the mode expansion to obtain the Hamiltonian in terms of modes.

3 Write down the action for a string coupled to the antisymmetric tensor field $B_{\mu\nu}$ and the dilaton field ϕ. What is the justification for calling $\exp\phi$ the string coupling constant? Write out the Polyakov path integral. Describe why one needs to introduce "ghost" fields. For a Riemann surface of genus g, describe what is meant by: (i) the moduli; (ii) the mapping class group (also sometimes called the modular group) and (iii) the conformal Killing group. Give a detailed description without mathematical technicalities of how these quantities enter into string scattering calculations.

The infinitesimal transformation of the metric under diffeomorphisms and Weyl transformation is given by:

$$\delta g_{\mu\nu} = (2\delta\omega - \nabla_\rho \delta\sigma^\rho) g_{\mu\nu} - 2(P_1\delta\sigma)_{\mu\nu} \,,$$

where

$$(P_1\delta\sigma)_{\mu\nu} = \frac{1}{2}(\nabla_\mu \delta\sigma_\nu + \nabla_\nu \delta\sigma_\mu - g_{\mu\nu}\nabla_\rho\delta\sigma^\rho) \,.$$

Calculate the conditions for conformal Killing vectors and the equation obeyed by infinitesimal moduli variations.

For a genus 1 world sheet (that is the torus) both the conformal Killing group and the moduli space have complex dimension 1. Evaluating a scattering process involving four tachyon vertex operators on the torus would require carrying out how many integrations, and why?