UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 7 June, 2004 1.30 to 4.30

PAPER 48

THE STANDARD MODEL

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a scalar field theory with a scalar with components ϕ_r. The potential for the field $V(\phi)$ is invariant under infinitesimal transformations

$$\delta\phi = iT_a\chi_a\phi\,, \quad a = 1, \ldots \dim G\,,$$

where T_a are the $\dim G$ generators of invariance group G in the representation defined by ϕ and χ_a are some infinitesimal parameters. The potential has a degenerate vacuum labelled by Φ_0. t_i are the generators of H, which is the stability group for $\phi_0 \in \Phi_0$, i.e.

$$t_i\phi_0 = 0\,, \quad i = 1, \ldots \dim H\,.$$

Choosing a basis for the generators such that

$$T_a = (t_i, T_{\hat{a}})\,,$$

with $T_{\hat{a}}$ orthogonal to t_i, prove, by expanding about the vacuum ϕ_0, that there are $\dim G - \dim H$ massless scalars.

If $G = O(n)$, the rotation group in n dimensions, and $H = O(n-1)$ then how many massless scalars are there in this case?

The Lagrangian for the gauge-scalar sector of the standard model may be written as,

$$\mathcal{L} = -\frac{1}{4}\,\mathbf{F}^{\mu\nu}\cdot\mathbf{F}_{\mu\nu} - \frac{1}{4}\,G^{\mu\nu}G_{\mu\nu} + (D^\mu\phi)^\dagger D_\mu\phi - \frac{1}{2}\lambda\big(\phi^\dagger\phi - \frac{v^2}{2}\big)^2,$$

where

$$\mathbf{F}_{\mu\nu} = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + g\,\mathbf{A}_\mu \times \mathbf{A}_\nu \quad , \quad G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$
$$(D^\mu\phi) = (\partial^\mu + ig\tfrac{1}{2}\mathbf{A}^\mu(\mathbf{x})\cdot\sigma + i\tfrac{1}{2}g'B^\mu(x))\phi,$$

and $\mathbf{A}_\mu$ is the vector of $SU(2)$ gauge fields, B_μ is the $U(1)_Y$ gauge field and ϕ is a complex scalar doublet. σ_i are the Pauli matrices and g' may be written as $g\tan\theta_W$.

Explain why the scalar doublet can be written as

$$\phi(x) = \exp(-i\mathbf{n}(x).\sigma + in_3(x))\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v + H(x) \end{pmatrix},$$

where $\mathbf{n} = (n_1, n_2, n_3)$, and why in unitary gauge we can eliminate the fields in $\exp(-i\mathbf{n}(x).\sigma + in_3(x))$ completely.

Determine the simultaneous mass and charge eigenstates for the gauge fields by writing the scalar-boson interactions in terms of the physical fields Z_μ, $W^\pm_\mu$, and show that the photon field A_μ decouples from the scalar and is massless. Find also the mass of the scalar field and the the relationship between the masses m_W and m_Z.

2 The covariant derivative in the electroweak sector of the Standard Model is defined by

$$D^\mu = (\partial^\mu + ig\tfrac{1}{2}\mathbf{A}^\mu(\mathbf{x})\cdot\sigma + iYg'B^\mu(x)),$$

where g is the $SU(2)$ coupling constant, g' is the $U(1)_Y$ coupling constant, σ_i are the Pauli matrices and Y is the hypercharge. Describe the $SU(2)$ and hypercharge representations and quantum numbers for the leptons. Using

$$W^{+\mu} = \frac{1}{\sqrt{2}}(A_1^\mu - iA_2^\mu), \qquad W^{-\mu} = \frac{1}{\sqrt{2}}(A_1^\mu + iA_2^\mu),$$

show that the interaction of the W^+ boson with the lepton fields may be written as

$$\mathcal{L}_{W+lep} = -\frac{g}{2\sqrt{2}}W^{+\mu}\bar{\nu}_l\gamma_\mu(1-\gamma^5)l,$$

for each family of leptons. Write the corresponding term for the W^- boson. Explain briefly why the corresponding interaction term is more complicated for quarks.

Show that at low energies it is equivalent to using an effective Lagrangian density

$$\mathcal{L}_{\text{Weff}} = -\frac{G_F}{\sqrt{2}}(J^\mu(x)^\dagger J_\mu(x))\,,$$

where $G_F = \sqrt{2}g^2/8m_W^2$.

Consider the decay

$$\pi^-(p) \to e^-(k) + \overline{\nu}_e(q).$$

The matrix element for this decay is

$$\mathcal{M} = -\tfrac{G_F}{\sqrt{2}}\langle e^-(k)\,\overline{\nu}_e(q)|\bar{e}\gamma^\alpha(1-\gamma_5)\nu_e|0\rangle\,\langle 0|J_\alpha^{\text{had.}}|\pi^-(p)\rangle.$$

Explain why only the axial part of the hadronic current contributes to $\langle 0|J_\alpha^{\text{had.}}|\pi^-(p)\rangle$.

By considering this matrix element prove that the decay rate contains a factor $m_e^2(m_\pi^2 - m_e^2)$, and hence vanishes in the limit $m_e \to 0$. (It is useful to use the Dirac equation for the spinors in momentum space.) Explain physically why the matrix element must vanish in this limit.

[You may use

$$\text{tr}\,\{\gamma.k\gamma.q\} = 4k\cdot q$$
$$\text{tr}\,\{\gamma^5\gamma.k\gamma.q\} = 0$$
$$\text{tr}\,\{\gamma^\mu\} = \text{tr}\,\{\gamma^5\gamma^\mu\} = 0.]$$

3 Under charge conjugation we assume

$$\psi(x) \longrightarrow \psi^C(x) \ , \quad \psi^C(x) = C\overline{\psi}(x)^t \ ,$$

with t denoting transpose. The matrix C is then chosen to ensure $\psi^C(x)$ satisfies the Dirac equation. Prove that $C(\gamma^\mu)^t C^{-1} = -\gamma^\mu$. Show also that under charge conjugation $\bar{\psi}(x) \to -\psi^t(x)C^{-1}$. (Assume $C^\dagger = C^{-1}$.)

Explain why a current interaction

$$J^\mu V_\mu = \bar{\psi}(x)\gamma^\mu(1-\gamma^5)\psi(x)V_\mu,$$

is not invariant under parity or charge conjugation separately but is invariant under the combined transformation.

Under a time reversal transformation $\hat{T}\psi(x)\hat{T}^{-1} = B^{-1}\psi(x_T)$ and $\hat{T}\overline{\psi}(x)\hat{T}^{-1} = \overline{\psi}(x_T)B$ where $B = \gamma_5 C$ and $\hat{T}$ is an antilinear transformation, i.e. it takes the complex conjugate of numbers. Show that $B(\gamma^{0*}, -\gamma^*)B^{-1} = (\gamma^0, \gamma)$, and that the above current interaction is invariant under time reversal.

The K^0 and its anti-particle $\bar{K}^0$ are pseudoscalar mesons with dominant quark structure $\bar{s}d$ and $\bar{d}s$. Under CP we can define

$$\hat{C}\hat{P}|K^0\rangle = |\bar{K}^0\rangle \, , \qquad \hat{C}\hat{P}|\bar{K}^0\rangle = |K^0\rangle \, .$$

The mass eigenstates of the system are the eigenvectors of the matrix

$$M = \begin{pmatrix} \langle K^0|H'|K^0\rangle & \langle K^0|H'|\bar{K}^0\rangle \\ \langle \bar{K}^0|H'|K^0\rangle & \langle \bar{K}^0|H'|\bar{K}^0\rangle \end{pmatrix} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} ,$$

where H' is an effective Hamiltonian arising from weak processes that mix $|K^0\rangle$ and $|\overline{K}^0\rangle$, and $M_{11} = M_{22}$. Draw a Feynman diagram representing one such mixing process. Show that if H' is not invariant under CP then $M_{12} \neq M_{21}$ and that the mass eigenstates are equal to the $CP = +1$ and -1 eigenstates

$$|K_1{}^0\rangle = \frac{1}{\sqrt{2}}\Big(|K^0\rangle + |\bar{K}^0\rangle\Big) \, , \qquad |K_2{}^0\rangle = \frac{1}{\sqrt{2}}\Big(|K^0\rangle - |\bar{K}^0\rangle\Big) \, ,$$

up to small corrections proportional to

$$\epsilon = \frac{\sqrt{M_{12}} - \sqrt{M_{21}}}{\sqrt{M_{12}} + \sqrt{M_{21}}}.$$

[Under parity transformations P

$$\psi(x) \to \gamma^0\psi(x_P) \qquad \bar{\psi}(x) \to \bar{\psi}(x_P)\gamma_0 \qquad V_\mu(x) \to V^\mu(x_P).$$

Under charge conjugation C $V_\mu(x) \to -V_\mu(x)$, and under time reversal $V_\mu(x) \to V^\mu(x_T)$]

4 Show that the total cross-section for $e^-(p_1) + e^+(p_2) \rightarrow \gamma^*(p_1 + p_2 = q) \rightarrow q(k_1) + \overline{q}(k_2)$ at lowest order is equal to

$$\frac{\mathrm{d}\sigma_{e^-e^+\rightarrow q\bar{q}}}{\mathrm{d}\Omega} = \frac{\alpha^2}{4q^2}\, Q_q^2\, (1 + \cos^2\theta)\,,$$

where $\alpha = e^2/4\pi$, θ is the angle between the outgoing quark and the axis of the incoming electron and positron in the centre of mass frame, and Q_q is the fractional quark charge. Show that integrating over the solid angle

$$\sigma_{e^-e^+\rightarrow q\bar{q}} = \frac{4\pi\alpha^2}{3q^2}\, Q_q^2\,.$$

You may assume that $\sqrt{q^2} \gg m_q, m_e$.

Explain why at leading order this means that (to a good approximation)

$$\sigma_{e^-e^+\rightarrow \mathrm{hadrons}} = \frac{4\pi\alpha^2}{3q^2}\, 3\sum_f Q_f^2\,,$$

Discuss why beyond leading order the cross-section for quark-antiquark production is not a well-defined physical quantity, and how one may calculate the total hadron cross-section.

Beyond LO the cross-section may be written as

$$\sigma_{e^-e^+\rightarrow \mathrm{hadrons}} = \frac{4\pi\alpha^2}{3q^2}\, 3\sum_f Q_f^2\, K(\alpha_s(\mu^2), q^2/\mu^2)\,,$$

where at $\mathcal{O}(\alpha_s^2)$

$$K(\alpha_s(\mu^2), q^2/\mu^2) = 1 + \frac{\alpha_s(\mu^2)}{\pi} + \frac{\alpha_s^2(\mu^2)}{\pi^2}\left(1.99 - 0.11 n_f - \pi\frac{\beta_0}{4\pi}\ln(q^2/\mu^2)\right).$$

One way of choosing the arbitrary scale μ is to demand that

$$\frac{dK(\alpha_s(\mu^2), q^2/\mu^2)}{d\ln\mu^2} = 0.$$

Using the renormalization group equation for the strong coupling

$$\frac{d\alpha_s}{d\ln\mu^2} = -\frac{\beta_0}{4\pi}\alpha_s^2,$$

where $\beta_0 = 11 - 2/3 n_f$, and n_f is the number of quark flavours, determine the value of μ^2 this prescription imposes.

[You may use $\mathrm{tr}\,(\gamma_\alpha\gamma_\beta\gamma_\gamma\gamma_\delta) = 4(g_{\alpha\beta}g_{\gamma\delta} + g_{\alpha\delta}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\delta})$, and

$$\sigma = \frac{1}{4F}\frac{1}{4}\sum_{\mathrm{spins}}\int \frac{d^3\mathbf{k_1}}{(2\pi)^3 2E_{\mathbf{k_1}}}\frac{d^3\mathbf{k_2}}{(2\pi)^3 2E_{\mathbf{k_2}}}(2\pi)^4\delta^4(p_1 + p_2 - k_1 - k_2)|M|^2$$

where the flux factor $F = 4\sqrt{(p_1.p_2)^2 - m_1^2 m_2^2} = 2q^2$, where we let $m_1, m_2 \rightarrow 0$.]