UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 2 June, 2004 1.30 to 4.30

PAPER 47

ADVANCED QUANTUM FIELD THEORY

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A scalar field ϕ in four dimensions has classical action S. Write down equations defining the generating functional for correlation functions, Z, and the generating functional for connected correlation functions, W, in terms of a functional integral. Use the relation between Z and W (which you need not justify) to express the full four-point function in terms of connected correlation functions, assuming $\delta W/\delta J = 0$ when $J = 0$.

Describe carefully how to construct the effective action, Γ, from W. Given that

$$i\Gamma[\phi] \;=\; \sum_{n\geqslant 2} \frac{1}{n!} \int d^4x_1 \ldots \int d^4x_n \, \phi(x_1) \ldots \phi(x_n) \, \Gamma_n(x_1, \ldots, x_n) \;,$$

derive expressions for Γ_2 and Γ_3 in terms of connected correlation functions.

Now assume that

$$W[J] = -\frac{1}{2} \int d^4x \int d^4y \, J(x) \Delta_F(x-y) J(y) \,,$$

where Δ_F is the Feynman propagator. Calculate Γ and hence show that $\Gamma_n = 0$ for $n \geqslant 3$. How do you interpret this result?

2 **(a)** Derive a formula for the superficial degree of divergence of an amputated, one-particle-irreducible (1PI) Feynman graph with E external lines in ϕ^4 theory in four spacetime dimensions. Give an example of a 1PI graph that is divergent and yet superficially convergent. State clearly the form of the counterterms, and the Feynman rules which correspond to them, in renormalized perturbation theory. Explain, in outline, how this approach leads to finite expressions for correlation functions and why it is the *superficial* degree of divergence which is important. [You should not attempt evaluation of any Feynman graphs.]

(b) A bosonic field Φ in d spacetime dimensions has a kinetic term of the form $(\partial\Phi)^2$ and an interaction term of the form $\lambda\Phi^n(\partial\Phi)^r$, where ∂ denotes a spacetime derivative but all details of Lorentz indices and how they are contracted may be ignored. Consider the momentum space Feynman rules for this theory: each diagram consists of lines, which represent Φ propagators, joining vertices, which represent the interaction. How many lines meet at a vertex, and what is the power of momentum associated with each vertex? Show that the superficial degree of divergence of a 1PI diagram is

$$d \;-\; \frac{d-2}{2} E \;-\; V \dim(\lambda)$$

where V is the number of vertices, E is the number of external lines and $\dim(\lambda)$ is the mass dimension of the coupling, which you should determine as a function of n, r and d.

(c) Write down the self-interaction terms which arise from the Yang-Mills lagrangian for a nonabelian gauge field A^a_μ. Is the result in part (b) consistent with the statement that Yang-Mills theory is renormalizable in $d = 4$ but not in $d = 6$; if so, why?

3 A scalar field ϕ has mass m and interaction lagrangian $-\lambda\phi^4/4!$ in four dimensions. Assuming standard Feynman rules, draw the one-loop, amputated, one-particle irreducible graphs contributing to the four-point function with fixed external momenta. Evaluate these diagrams using dimensional regularization, expressing the finite parts as integrals over a Feynman parameter (which you should not attempt to carry out).

Using minimal subtraction, calculate the one loop beta-function for this theory. Explain, very briefly, any difference that arises in the corresponding result for a non-abelian gauge theory without matter fields, and why this is significant.

$$\left[\quad \Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \sim \frac{1}{z} - \gamma_E + O(z) \qquad \text{as} \qquad z \to 0 \ . \quad\right]$$

4 A non-abelian gauge group has anti-hermitian Lie algebra generators T_a obeying

$$[T_a, T_b] = f_{abc} T_c \ , \qquad \mathrm{Tr}(T_a T_b) = -\delta_{ab} \ .$$

Describe clearly how the gauge field A^a_μ behaves under a gauge transformation $U(x^\mu)$ taking values in the Lie group. Define the field strength $F^a_{\mu\nu}$ and state how it behaves under the same gauge transformation. Derive expressions for the changes in the gauge field and field strength under an infinitesimal transformation $U(x^\mu) = 1 + e\,\omega_a(x^\mu) T_a$, where terms of second or higher order in ω_a can be neglected (e is the coupling).

Explain the difficulties which arise in quantizing such a gauge theory, assuming it is governed by a gauge-invariant action $S[A^a_\mu]$, and give an account of how these difficulties can be resolved. Statements regarding functional integrals may be justified by comparison with appropriate, finite-dimensional results; these finite-dimensional results need not be proved but must be accurately quoted. Your account should include explanations of the terms *gauge-fixing*, *Faddeev-Popov determinant*, and *ghosts*.

The BRST symmetry of the gauge-fixed action is generated by a hermitian charge Q in the operator formalism. Explain how Q is used to define physical states in the quantum theory and how this definition depends on a certain key property of the charge. State, without proof, what this general definition implies for the polarization ε_μ of a physical photon state with momentum k_μ (where $k_\mu k^\mu = 0$).