UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 31 May, 2004 1.30 to 4.30

PAPER 45

SYMMETRY AND PARTICLE PHYSICS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the Clebsch Gordan coefficients $\langle j_1 m_1\, j_2 m_2 | JM\rangle$. Explain why we expect, for fixed M,

$$\sum_{m_1,m_2} \langle j_1 m_1\, j_2 m_2 | JM\rangle^2 = 1\,.$$

Particles belonging to an isospin multiplet with isospin I decay under strong interactions to two particles with isospins I_1 and I_2. Denoting the particles by (IM), where I, M are their isospin quantum numbers, explain why the amplitude for the decay should have the form

$$A_{(IM)\to(I_1M_1)+(I_2M_2)} = a\, \langle I_1 M_1\, I_2 M_2 | IM\rangle\,,$$

where a is independent of M_1, M_2, M.

Show that the total decay rate $\Gamma_{(IM)\to \text{anything}} = |a|^2$.

Consider the decay of the Δ baryons to a proton p or neutron n and a pion π. Using standard assignments of isospins work out

$$\frac{\Gamma_{\Delta^+\to p+\pi^0}}{\Gamma_{\Delta^{++}\to p+\pi^+}}\,, \qquad\qquad \frac{\Gamma_{\Delta^+\to n+\pi^+}}{\Gamma_{\Delta^{++}\to p+\pi^+}}\,.$$

How might Δ^- decay?

[Note that $I_-|I\,M\rangle = \sqrt{(I+M)(I-M+1)}|I\,M{-}1\rangle$.]

2 The quarks $q^\alpha = (u,d,s)$, for $\alpha = 1,2,3$, transform under the 3 dimensional representation of $SU(3)$. Describe briefly how the conjugate $\bar q_\alpha = (\bar u, \bar d, \bar s)$ transforms under the conjugate 3^* representation.

Show how we may decompose the tensor products of representations such that

$$3\times 3 = 3^* + 6\,, \qquad 3\times 3\times 3 = 1+8+8+10\,, \qquad 3^*\times 3^*\times 3 = 3^* + 3^* + 6 + 15\,,$$

where representations are labelled by their dimensions and 15 denotes the representation formed by tensors $T^\alpha_{\beta\gamma}$ satisfying $T^\alpha_{\beta\gamma} = T^\alpha_{\gamma\beta}$, $T^\alpha_{\beta\alpha} = 0$. How does $3^*\times 3^*\times 3^*$ decompose?

A possible baryon has strangeness $S = 1$ and isospin $I = 0$. Show how this might be interpretated as belonging to a 10^* representation with quark structure $(ud)(ud)\bar s$ where (ud) denotes u, d quarks combined so that they belong to a 3^* representation. Why can this baryon not belong to an 8 representation?

3 Show that the parity of a meson formed from a quark and antiquark is $(-1)^{L+1}$, where L is their relative orbital angular momentum in the centre of mass frame.

Show that the charge conjugation of a neutral meson is $(-1)^{L+S}$ where S is the total spin of the quark, antiquark.

Restricting to the u, d quarks and their antiquarks show that we may expect amongst low mass mesons the spinless pions $\pi^\pm, \pi^0$ and also spin 1 mesons $\rho^\pm, \rho^0$, with isosopin 1, and ω with isospin 0. What are their C, P quantum numbers? Why can $\rho^0 \to \pi^+\pi^-$ but $\omega \not\to \pi^+\pi^-$ although $\omega \to \pi^+\pi^-\pi^0$?

[You may assume that the parity of an antiquark is opposite to that of the corresponding quark and that charge conjugation interchanges quarks and antiquarks.]

4 What is the rank of a Lie algebra? Describe briefly how for a Lie algebra $\mathcal{L}$ the roots $\boldsymbol{\alpha}$ may be defined.

Let H, $E^\pm$ be elements of $\mathcal{L}$ satisfying

$$[H, E^\pm] = \pm 2E^\pm \,, \qquad [E^+, E^-] = H \,,$$

and let $\{X_0, X_1, \ldots\} \in \mathcal{L}$ be such that

$$[E^-, X_0] = 0 \,, \qquad [H, X_0] = -\lambda X_0 \,, \qquad [E^+, X_n] = X_{n+1} \,.$$

What is $[H, X_n]$? Assume

$$[E^-, X_n] = q_n X_{n-1} \,.$$

By considering the commutator of this equation with E^+ and using the Jacobi identity show that $q_{n+1} = q_n + \lambda - 2n$ and hence that

$$q_n = n(\lambda - n + 1) \,.$$

Suppose that $[E^+, X_{n_0}] = 0$ for some n_0. Show that we must then have $\lambda = n_0$.

Let H_i, $E_i^\pm \in \mathcal{L}$ for $i = 1, 2$, where

$$[H_i, H_j] = 0 \,, \quad [E_i^+, E_j^-] = \delta_{ij} H_j \,, \quad [H_i, E_j^\pm] = \pm K_{ji} E_j^\pm] \,, \quad K_{jj} = 2 \,, \quad \text{no sum on } j \,.$$

Explain why

$$[\underbrace{E_1{}^\pm, [\ldots [E_1{}^\pm}_{n}, E_2{}^\pm] \ldots]] \neq 0 \,, \quad [\underbrace{E_1{}^\pm, [\ldots [E_1{}^\pm}_{n+1}, E_2{}^\pm] \ldots]] = 0 \quad \Rightarrow \quad K_{21} = -n \,.$$

If $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ are roots why must $2\,\boldsymbol{\alpha} \cdot \boldsymbol{\beta} / \boldsymbol{\beta}^2$ be an integer?

$R^r{}_s$, $\sum_r R^r{}_r = 0$, $r, s = 1, 2, 3$ are elements of a Lie algebra such that

$$[R^r{}_s, R^u{}_v] = \delta^u{}_s \, R^r{}_v - \delta^r{}_v \, R^u{}_s \,.$$

Show that we may take $E_1{}^+ = R^1{}_2$, $E_2{}^+ = R^2{}_3$ and $E_1{}^- = R^2{}_1$, $E_2{}^- = R^3{}_2$. What are H_1, H_2? Determine K_{ji} in this case.