UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 27 May, 2004 9:00 to 12:00

PAPER 44

QUANTUM FIELD THEORY

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the Lorentz group G. By considering Lorentz transformations close to the identity, show that there is a subgroup G_0 of Lorentz transformations whose elements L have the form

$$L = \exp\Big(\frac{1}{2}\omega_{\rho\sigma}M^{\rho\sigma}\Big)$$

where $\omega_{\rho\sigma}$ are parameters and $M^{\rho\sigma}$ are a set of constant 4×4 matrices. How many independent parameters are there?

The matrices $M^{\rho\sigma}$ may be chosen so that they satisfy the commutation relations

$$[M^{\rho\sigma}, M^{\tau\lambda}] = g^{\sigma\tau}M^{\rho\lambda} - g^{\rho\tau}M^{\sigma\lambda} + g^{\rho\lambda}M^{\sigma\tau} - g^{\sigma\lambda}M^{\rho\tau}$$

where $g^{\sigma\tau}$ is the Minkowski metric tensor. [You need not prove this.] Explain how one constructs a representation of these commutation relations, and hence of the group G_0, acting on Dirac spinors.

Discuss the way that the requirement of Lorentz invariance constrains the structure of Lagrangian densities in scalar field theory. In a theory coupling a scalar field ϕ to a Dirac spinor field ψ, three possible terms in the Lagrangian density are $\partial_\mu\phi\, \partial^\mu\phi$, $\partial_\mu\phi\, j_5^\mu$ and $j_{5\mu}\, j_5^\mu$, where $j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$ is the axial current. Explain, giving your reasons, whether these terms are compatible with Lorentz invariance.

2 State Noether's theorem for a Lagrangian field theory whose only fundamental field is a Lorentz 4-vector $A_\mu(x)$.

Consider pure electromagnetism, with A_μ the gauge potential. How does A_μ change under a gauge transformation? Define the field tensor $F_{\mu\nu}$ and show that the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

is gauge invariant. Why does a minus sign appear here?

Show that $A_\mu \to A_\mu + \alpha\partial_0 A_\mu$ (with α infinitesimal) is an infinitesimal symmetry of $\mathcal{L}$, and use Noether's theorem to find the conserved energy of the theory. An alternative infinitesimal symmetry is $A_\mu \to A_\mu + \alpha(\partial_0 A_\mu - \partial_\mu A_0)$. Use this to find an expression for the energy density which is gauge invariant. Comment on the interpretation of the extra term $-\alpha\partial_\mu A_0$.

3 Write brief notes on

a) Plane wave solutions of the Dirac equation,

b) Anticommutation relations for Dirac fields,

c) The particle number operator for a Dirac field,

d) Positrons.

[The Heisenberg field operator in the Dirac theory has the expansion

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2E_{\mathbf{p}}}} \sum_s \left(a^s_{\mathbf{p}} u_s(p) e^{-ip\cdot x} + b^{s\dagger}_{\mathbf{p}} v_s(p) e^{ip\cdot x} \right)$$

with $p_0 \equiv E_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$.]

4 Use the Feynman rules of QED to find the amplitude, to second order in the coupling e, for photon-electron elastic scattering. State carefully any conditions that the external photon momenta and polarizations should satisfy. [The complete QED Feynman rules need not be stated, only those relevant to the process here.]

Show that if the incoming photon has 4-momentum k and polarization 4-vector ϵ, then the scattering amplitude is unaffected if ϵ is replaced by $\epsilon + \alpha k$ (α a real constant). Discuss briefly the significance of this observation.

Sketch the diagrams, if any, that can contribute to the amplitude at third or fourth order in e. [You need not evaluate the diagrams.]