UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 3 June, 2004 1.30 to 4.30

PAPER 4

PRO-p GROUPS

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define an *inverse system* and an *inverse limit* of topological spaces.

Prove that inverse limits exist and are unique.

Hence explain why the two definitions of a profinite group are equivalent i.e. if G is a topological group then G is compact and totally disconnected if and only if G is an inverse limit of finite groups (state results from topology as required).

Let G be a profinite group and let p be a prime. For N an open normal subgroup of G denote by $\mathcal{P}(N)$ the set of Sylow p-subgroups of G/N. By considering a suitable inverse system of finite sets show that G has a closed subgroup P such that $PN/N \in \mathcal{P}(N)$ for every open normal subgroup N. Show that P is a maximal pro-p subgroup of G (i.e. if Q is another pro-p subgroup of G and $P \leqslant Q$ then $P = Q$).

2 Define a *topological group*. Prove the following results.

(i) Every open subgroup of a topological group is closed.

(ii) The closure of any subgroup of a topological group is also a subgroup.

(iii) Every open subgroup of a profinite group has finite index and contains an open normal subgroup.

(iv) Every open subgroup of a finitely generated profinite group is finitely generated.

Let G be a d-generator pro-p group. Show that every generating set for G contains a subset of size at most d that generates G. (Results about the Frattini subgroup of G can be used without proof but should be clearly stated.)

3 Let G be a finite p-group for p an ODD prime and N a subgroup of G. Explain what it means to say that N is *powerfully embedded* in G and that G is *powerful*.

Prove that if N is powerfully embedded in G then N^p is powerfully embedded in G, and $\langle N, x\rangle$ is powerful for any $x \in G$. (You may use the fact that if M is powerfully embedded in G modulo $[M, G, G]$ then M is powerfully embedded in G.) Define the *lower p-series* $G_1 \geqslant G_2 \geqslant G_3 \geqslant \cdots$ of G, and prove that if G is powerful and $G = \langle a_1, \ldots, a_n\rangle$ then the following results hold.

(i) For each $i \geqslant 1$, G_i is powerfully embedded in G and $G_{i+1} = G_i^p = \Phi(G_i)$.

(ii) The map $x \mapsto x^p$ induces a homomorphism from G/G_2 onto G_2/G_3.

(iii) Every element of G^p is a p-th power in G, and $G^p = \langle a_1^p, \ldots, a_n^p\rangle$.

(iv) For each $i \geqslant 1$, $G_{i+1} = \langle a_1^{p^i}, \ldots, a_n^{p^i}\rangle$.

(v) $G = \langle a_1\rangle\langle a_2\rangle \cdots \langle a_n\rangle$.

Conversely show that if G is a product of cyclic groups then G is powerful.

4 Let p be an ODD prime and G a finitely generated powerful pro-p group. Explain what it means to say that G is *uniform* and prove that G is uniform if and only if it is torsion-free.

Given G a uniform pro-p group define the intrinsic Lie algebra L_G. Now define a powerful Lie algebra L and explain how to define a uniform pro-p group $(L, *)$.

Show that the assignments $G \mapsto L_G$ and $L \mapsto (L, *)$ satisfy the following:

(a) $L_{(L,*)} = \mathrm{L}$

(b) $(L_G, *) = \mathrm{G}$.

5 Write an essay describing the proof of the linearity of uniform pro-p groups.