UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 31 May, 2004 9:00 to 12:00

PAPER 34

Advanced Financial Models

Attempt **FOUR** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a one-period model in which there are $s+1$ assets of which exactly one is riskless and the remainder are risky. Define (i) an *arbitrage* and (ii) an *equivalent martingale probability.*

Prove that there is no arbitrage if and only if there exists an equivalent martingale probability.

Suppose now that the underlying probability space is finite and that there is no arbitrage. Prove that the market is complete if and only if there is a unique equivalent martingale probability.

[*You may quote the Separating Hyperplane Theorem without proof but any other result that you use should be proved carefully.*]

2 Write an essay on the standard binomial model operating over the times $0, 1, \ldots, n$ $(n \geqslant 2)$ where the stock price at time r is denoted by S_r, and between each period the stock price has a proportional up jump u or down jump d. Explain carefully how contingent claims are priced within the model.

Let $g_r(S_r)$ represent the price at time r of a claim which pays $f(S_n)$ at time n. When f is convex show that g_r is convex on the possible values that S_r can take on (viz. $S_r = S_0 u^i d^{r-i}$, $i = 0, 1, \ldots, r$).

Show that when f is convex then the amount of stock held in the hedging portfolio increases between the times r and $r+1$ $(< n)$ if the stock price increases between r and $r+1$.

[Here, interpret 'increase' in the weak sense, unless f is *strictly* convex.]

3 State Girsanov's Theorem and give a sketch of its proof.

For a standard Brownian motion $\{W_s,\ 0 \leqslant s \leqslant t\}$ calculate

$$\mathbb{P}\left(\sup_{0 \leqslant s \leqslant t} (W_s + \mu s) \leqslant a\right),$$

for real μ and $a > 0$.

In the Black-Scholes model consider an up-and-out claim that delivers $\left(S_{t_0}\right)^2$ at time t_0 if the stock price $\{S_t,\ t \geqslant 0\}$ stays below a pre-determined level $c > S_0$ between time 0 and time t_0, otherwise it pays nothing. Determine the price at time 0 of this claim as a function of c, t_0, ρ and σ where σ is the volatility of the stock and ρ is the fixed continuously-compounded interest rate.

4 In the context of the Black-Scholes model, let S_t be the price at time t of the stock, σ its volatility and let ρ be the fixed continuously-compounded interest rate. For some fixed t_0, let $f(x,t)$ be a function of the form

$$f(x,t) = g(x,t)x + h(x,t)e^{-\rho(t_0-t)}.$$

Derive necessary and sufficient conditions on the functions g and h so that the portfolio holding $g(S_t,t)$ in stock and $h(S_t,t)$ in the bond maturing at time t_0 is self-financing.

(You may assume any smoothness conditions on g and h that you require.)

Deduce that $f(S_t,t)$ is the value of a self-financing portfolio if and only if f satisfies

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2} + \rho x \frac{\partial f}{\partial x} + \frac{\partial f}{\partial t} - \rho f = 0.$$

Determine explicitly the self-financing portfolio that always maintains a fixed proportion γ, $0 < \gamma < 1$, of its value in its holding in stock.

5 An investor in the Black-Scholes model has initial wealth w_0 and wishes to invest so as to maximize the expected utility of his final wealth at time t_0, where his utility function $u(\cdot)$ is concave and differentiable. Explain how he should determine his investment strategy and illustrate the solution in the case when $u(x) = \left(1 - e^{-ax}\right)/a$ for some constant $a > 0$.

Show how the solution may be extended to cover the case where the investor gains utility at the rate $v(R_t)$ per unit time when he consumes wealth at rate R_t as well as gaining utility from his final accumulated wealth. Solve this problem in the case when $u(x) = a\ln(x)$ and $v(x) = b\ln(x)$.

6 Give a short description of the modelling of bond prices in terms of instantaneous forward interest rates and show that the discounted bond prices are martingales if and only if the bond prices $\{P_{s,t},\ 0 \leqslant s \leqslant t\}$ may be represented in terms of the short-rate process $\{R_s,\ s \geqslant 0\}$ as

$$P_{s,t} = \mathbb{E}\left[e^{-\int_s^t R_u du} \,\middle|\, \mathcal{F}_s \right], \quad \textit{for all} \quad 0 \leqslant s \leqslant t.$$

(You should explain carefully all the terminology and notation used.)

Now consider the one-factor Vasicek model where

$$dR_s = \alpha\left(\beta - R_s\right) ds + \sigma dW_s$$

for constants α, β and σ and $\{W_s,\ s \geqslant 0\}$ is a standard Brownian motion. Assume that $\mathcal{F}_s = \sigma\left(W_u,\ 0 \leqslant u \leqslant s\right) = \sigma\left(R_u,\ 0 \leqslant u \leqslant s\right)$. Show that when the discounted bond prices are martingales then the bond prices take the form

$$P_{s,t} = \exp\left[a_{s,t} - b_{s,t}R_s\right],$$

for suitable deterministic $a_{s,t}$ and $b_{s,t}$, which should be calculated explicitly.

Show that if β is replaced by a suitably-chosen deterministic function θ_s then this model may be made to fit any observed bond prices at time $s = 0$.