UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 2 June, 2004 1.30 to 3.30

PAPER 31

LARGE DEVIATIONS AND QUEUES

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

While rigorous answers are preferred, heuristic answers will still gain partial credit.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $(X_n,\ n \in \mathbb{N})$ satisfy a large deviations principle in some space $\mathcal{X}$ with good rate function I. Let f be a bounded continuous function $\mathcal{X} \to \mathbb{R}$.

(a) Let $C_1, \ldots, C_m$ be closed subsets of $\mathcal{X}$ with $\bigcup_i C_i = \mathcal{X}$. Prove that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{nf(X_n)} \leqslant \max_{1\leqslant i\leqslant m} \Big\{ \sup_{x\in C_i} f(x) - \inf_{x\in C_i} I(x) \Big\}.$$

(b) Let $f(\mathcal{X})$ be contained in the interval $[a,b]$. Pick any $\varepsilon > 0$ and define the closed intervals

$$D_i = [a + (i-1)\varepsilon, a + i\varepsilon], \quad i = 1, \ldots, \lceil (b-a)/\varepsilon \rceil.$$

Let $C_i = f^{-1}(D_i)$. Using your answer to part (a), or otherwise, prove that

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{nf(X_n)} \leqslant \sup_{x\in\mathcal{X}} \Big[f(x) - I(x) + \varepsilon \Big].$$

(c) Pick any $\hat{x} \in \mathcal{X}$ and any $\varepsilon > 0$. Define the open interval

$$D = (f(\hat{x}) - \varepsilon, f(\hat{x}) + \varepsilon).$$

Let $B = f^{-1}(D)$. Using this set, or otherwise, prove that

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{nf(X_n)} \geqslant f(\hat{x}) - I(\hat{x}) - \varepsilon.$$

(d) Deduce that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E} e^{nf(X_n)} = \sup_{x\in\mathcal{X}} \Big[f(x) - I(x) \Big].$$

2 A sequence of random variables $(X_n,\ n \in \mathbb{N})$ taking values in a metric space $\mathcal{X}$ is said to have *Hurstiness* $H \in (0,1)$ if the following three conditions are satisfied:

- $(X_n,\ n \in \mathbb{N})$ satisfies a large deviations principle with good rate function I at speed $n^{2(1-H)}$;
- there is some $\hat{x} \in \mathcal{X}$ such that $0 < I(\hat{x}) < \infty$;
- there is some $\mu \in \mathcal{X}$ such that $I(x) = 0$ only if $x = \mu$.

Suppose $(X_n,\ n \in \mathbb{N})$ has Hurstiness H. Let $G > H$, $G \in (0,1)$, and define the good rate function.

$$I'(x) = \begin{cases} 0 & \text{if } I(x) = 0 \\ \infty & \text{otherwise.} \end{cases}$$

(a) Prove that for any closed set C

$$\limsup_{n\to\infty} \frac{1}{n^{2(1-G)}} \log \mathbb{P}(X_n \in C) \leqslant -\inf_{x\in C} I'(x).$$

(b) Using your answer to (a), or otherwise, show that if D is an open set containing μ then

$$\mathbb{P}(X_n \notin D) \to 0.$$

Hence (or otherwise) show that for any open set E

$$\liminf_{n\to\infty} \frac{1}{n^{2(1-G)}} \log \mathbb{P}(X_n \in E) \geqslant -\inf_{x\in E} I'(x).$$

(c) Suppose that $(X_n,\ n \in \mathbb{N})$ has Hurstiness H, that $(Y_n,\ n \in \mathbb{N})$ has Hurstiness G, that X_n is independent of Y_n, and that both take values in $\mathbb{R}$. Show that $(X_n + Y_n,\ n \in \mathbb{N})$ has Hurstiness equal to the greater of H and G.

Note. You should mention any general results you use, but you need not state them formally. Recall that $(X_n,\ n \in \mathbb{N})$ is said to satisfy an LDP with rate function I and speed $n^{2(1-H)}$ if for all measurable sets $B \subset \mathcal{X}$

$$\begin{aligned} -\inf_{x\in B^\circ} I(x) &\leqslant \liminf_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathbb{P}(X_n \in B) \\ &\leqslant \limsup_{n\to\infty} \frac{1}{n^{2(1-H)}} \log \mathbb{P}(X_n \in B) \leqslant -\inf_{x\in \bar{B}} I(x)\,. \end{aligned}$$

3 Packets arrive at an Internet router as a Poisson process of rate λ packets per second. Each packet has a payload; payload sizes are independent of each other and of the arrival process, and have an exponential distribution with mean 1 kilobyte.

The router maintains two parallel queues, a 'payload queue' and a 'header queue'. When a packet arrives, the payload is stored in the former, and a packet header is stored in the latter. Packets are served in the order they arrive. The payload queue is served at constant rate C kilobytes per second, and when the entire payload of a packet has been served then that packet's header is removed from the header queue. Assume $C > \lambda$.

Both queues have finite space. The payload queue has space for 1000 kilobytes; the header queue has space for 1000 headers. As a queueing theorist, you are called in to advise on whether these are sensible choices.

(a) Let Q be the number of packet headers in the header queue. With reference to an $M/M/1$ queue (or otherwise), estimate the probability that $Q \geqslant q$. (For modelling purposes, you can treat both queues as having infinite space.)

(b) The payload queue may be modelled by a discrete-time queue, with timeslots of length δ, in which the number of packets arriving in each timeslot is a Poisson random variable with mean $\delta\lambda$, and the service offered in that timeslot is $C\delta$. Let R_δ be the amount of work in this discrete-time queue. Estimate the probability that $R_\delta \geqslant r$. (Again, for modelling purposes, you can treat both queues as having infinite space.)

(c) Which queue is more likely to overflow? Give an intuitive explanation for your answer.

Hint. If N is a Poisson random variable with mean λ then $\mathbb{E}t^N = e^{\lambda(t-1)}$. If X is an exponential random variable with mean λ^{-1} then $\mathbb{E}e^{\theta X} = \lambda/(\lambda - \theta)$.

4 Consider a queue operating in continuous time, with constant service rate C and finite buffer B, with arrival process $a \in \mathcal{C}_\mu$. It is known that if $\mu < C$ then the queue size at time 0 may be written as

$$\bar{q}(a) = \sup_{t \geqslant 0} \Big\{ \Big(\sup_{0 \leqslant s \leqslant t} x(-s,0] \Big) \wedge \Big(B + \inf_{0 \leqslant s \leqslant t} x(-s,0] \Big) \Big\}$$

where $x(-s,0] = a(-s,0] - Cs$ and $x \wedge y = \min(x,y)$. When $B = \infty$, denote this function by q. It is also known that $\bar{q}$ and q are continuous on $(\mathcal{C}_\mu, \|\cdot\|)$.

Suppose that $(A^L,\ L \in \mathbb{N})$ satisfies a large deviations principle in $(\mathcal{C}_\mu, \|\cdot\|)$ with good rate function

$$I(a) = \begin{cases} \int_{-\infty}^0 \Lambda^*(\dot{a}_s)\, ds & \text{if } a \text{ is absolutely continuous} \\ \infty & \text{otherwise} \end{cases}$$

for some strictly convex rate function Λ^* with $\Lambda^*(\mu) = 0$.

(a) Write down a large deviations principle for $q(A^L)$; let it have rate function J. Also write down a large deviations principle for $\bar{q}(A^L)$; let it have rate function $\bar{J}$.

(b) Show that $\bar{q}(a) \leqslant q(a)$. Hence (or otherwise) show that

$$\bar{J}(x) \geqslant \inf_{y \geqslant x} J(y).$$

(c) Show that J is increasing. Deduce that $\bar{J}(x) \geqslant J(x)$.

(d) Let $x \leqslant B$. Show that $\bar{J}(x) \leqslant J(x)$. *Hint. Let $\hat{a}$ be the most likely path to attain $q(a) = x$. What is $\bar{q}(\hat{a})$?*

(e) Deduce that $\bar{q}(A^L)$ satisfies a large deviations principle with good rate function

$$\bar{J}(x) = \begin{cases} J(x) & \text{if } x \leqslant B \\ \infty & \text{otherwise.} \end{cases}$$

Note. You may assume standard results about queues with infinite buffers.