UNIVERSITY OF CAMBRIDGE

# MATHEMATICAL TRIPOS Part III

Tuesday 1 June, 2004 9 to 11

## PAPER 29

## POISSON PROCESSES

*Attempt* **TWO** *questions.*

*There are* **three** *questions in total.*

*The questions carry equal weight.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Explain carefully what is meant by saying that the random sequence $(p_1, p_2, p_3, \ldots)$ has the Poisson-Dirichlet distribution with parameter $\theta$. Show that, for any polynomial $\phi$ with $\phi(0) = 0$,

$$\mathbb{E}\left\{\sum_{n=1}^{\infty} \phi(p_n)\right\} = \theta \int_0^1 \phi(x) x^{-1} (1-x)^{\theta-1} dx\,.$$

What does this tell you about the distribution of $p_1$?

**2** The positions of trees in a large forest can be modelled as a Poisson process $\Pi$ of constant density $\lambda$ on $\mathbb{R}^2$. Each tree produces a random number of seeds having a Poisson distribution with mean $\mu$. Each seed falls to earth at a point uniformly distributed over the circle of radius $r$ whose centre is the tree. The positions of the different seeds relative to their parent tree, and the numbers of seeds produced by a given tree, are independent of each other and of $\Pi$. Prove that, conditional on $\Pi$, the seeds form a Poisson process $\Pi^*$ whose mean measure depends on $\Pi$. Is the unconditional distribution of $\Pi^*$ that of a Poisson process? Justify your answer.

**3** A uniform Poisson process $\Pi$ in the unit ball of $\mathbb{R}^3$ is one whose measure is Lebesgue measure (volume) on $B = \{(x, y, z) \in \mathbb{R}^3 \ ; \ r^2 = x^2 + y^2 + z^2 \leqslant 1\}$. Show that

$$\Pi_1 = \{r \ ; \ (x, y, z) \in \Pi\}$$

is a Poisson process on $[0, 1]$ and find its mean measure. Show that

$$\Pi_2 = \{(x/r, y/r, z/r) \ ; \ (x, y, z) \in \Pi\}$$

is a Poisson process on the boundary of $B$, whose mean measure is a multiple of surface area. Are $\Pi_1$ and $\Pi_2$ independent processes? Justify your answer.