UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 4 June, 2004 1.30 to 4.30

PAPER 27

CYCLOTOMIC FIELDS

Attempt **FOUR** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

Notation: *Throughout p will denote an odd prime number, $\mathbb{Z}_p$ will denote the p-adic integers, and $\mathbb{Q}_p$ the field of p-adic numbers. For each $n \geqslant 0, \zeta_n$ will denote a primitive p^{n+1}-th root of unity, with the property that $\zeta_{n+1}^p = \zeta_n$ for all $n \geqslant 0$. Finally, K_n will denote the field $\mathbb{Q}_p(\zeta_n)$.*

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) Let $R = \mathbb{Z}_p[[T]]$ be the ring of formal power series in T with coefficients in $\mathbb{Z}_p$, and let $R^\times$ be the group of units of R. Prove that there is a unique map $N : R^\times \to R^\times$ such that

$$(Nf)((1+T)^p - 1) = \prod_{\zeta} f(\zeta(1+T) - 1)\,,$$

where the product on the right is taken over all p-th roots of unity ζ.

(b) Assume now that f is an element of $R^\times$ such that $Nf = f$. Put $u_n = f(\zeta_n - 1)$, and prove that $N_{K_n/K_{n-1}}(u_n) = u_{n-1}$ for all $n \geqslant 1$; here $N_{K_n/K_{n-1}}$ denotes the norm map from K_n to K_{n-1}.

2 (a) Let U_n be the group of units of K_n which are $\equiv 1$ modulo the maximal ideal, and define $U_\infty = \overset{\lim}{\longleftarrow} U_n$, where the projective limit is taken with respect to the norm maps. For each integer $k \geqslant 1$, define the logarithmic derivative maps

$$\delta_k : U_\infty \to \mathbb{Z}_p\,,$$

and determine their behaviour under the action of the Galois group on U_∞.

(b) Let a be any integer with $(a,p) = 1$, and let

$$c_n(a) = \left(\frac{\zeta_n^{-a} - \zeta_n^{a}}{\zeta_n^{-1} - \zeta_n}\right)^{p-1} \qquad (n = 0, 1, \ldots)\,.$$

Prove that $c(a) = (c_n(a))$ belongs to U_∞, and compute $\delta_k(c(a))$ for all $k \geqslant 1$.

3 Write an essay on the main conjecture on cyclotomic fields, including a discussion of Iwasawa's theorem and its relationship to the main conjecture.

4 Write an essay on the axiomatic theory of Euler systems for cyclotomic fields, illustrating the axiomatic theory by the Euler systems of cyclotomic units.