UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 2 June, 2004 9 to 11

PAPER 26

MODULAR FORMS

Attempt **ALL** *questions.*

There are **three** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 i) Define the spaces M_k, S_k of modular and cusp forms of weight k for the modular group $SL_2(\mathbb{Z})$. Assuming the dimension formula for M_k, show that every modular form with integral Fourier coefficients may be expressed as a polynomial in E_4, E_6 and Δ with integral coefficients.

[*You may assume any results about the Eisenstein series E_k and Δ that you require.*]

ii) Show further that there exists a unique basis for M_k consisting of forms g_j $(0 \leq j < d = \dim M_k)$ whose q-expansions are of the form

$$g_j = q^j + \sum_{n \geq d}^{\infty} c_n(j) q^n, \quad c_n(j) \in \mathbb{Z}.$$

Show also that if $f = \sum a_n q^n \in M_k$, then for all $n \geq d$,

$$a_n = \sum_{j=0}^{d-1} c_n(j) a_j.$$

iii) Let T_n be the n^{th} Hecke operator acting on S_k. Deduce that for every $n \geq 1$,

$$T_n = \sum_{j=1}^{d-1} c_n(j) T_j.$$

2 Define the Weierstrass $\wp$-function $\wp(z)$ associated to a lattice Λ. Show that it satisfies the differential equation $\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3$ where $g_2 = 60G_4(\Lambda)$, $g_3 = 140G_6(\Lambda)$ and

$$G_k(\Lambda) = \sum_{0 \neq \omega \in \Lambda} \frac{1}{\omega^k}.$$

Show also that if $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, and $\omega_3 = -\omega_1 - \omega_2$ then

$$\wp'(z)^2 = 4\prod_{i=1}^{3}(\wp(z) - e_i)$$

where $e_i = \wp(\omega_i/2)$, and that $e_i \neq e_j$ for $i \neq j$. Deduce that $\Delta(\Lambda) = g_2^3 - 27g_3^2$ is never zero.

3 Write an account of EITHER

(i) the theory of Hecke operators for modular forms on $SL_2(\mathbb{Z})$;

OR

(ii) the theory of the theta functions $\vartheta_{\alpha\beta}(z, \tau)$.