UNIVERSITY OF
CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 28 May, 2004 9 to 12

PAPER 25

CLASS FIELD THEORY

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Write an essay on Kummer theory. You should include a proof of Hilbert's theorem 90.

(ii) Let K be a number field, and let Δ be a finite subgroup of $K^*/(K^*)^n$. Determine, with proof, a finite set of places S, defined in terms of n and Δ, such that $K(\sqrt[n]{\Delta})$ is unramified outside S.

2 State and prove a version of Hensel's lemma. Use it to classify the unramified extensions of K where $[K : \mathbb{Q}_p] < \infty$. Use properties of the Herbrand quotient to show that if L/K is unramified then every unit in K is a norm from L.

3 (i) Explain what is meant by a place of a field K. State the weak approximation theorem.

(ii) Write an essay on idèles. Use them to show that the Artin reciprocity law is equivalent to the existence of certain local reciprocity maps.

4 Write an essay on central simple algebras and the Brauer group. You should include the construction of cyclic algebras and briefly indicate the role they play in computing the Brauer groups of finite fields, local fields and number fields.

5 (i) State the classification theorem of class field theory. Quoting any results you need about conductors, deduce the existence of the Hilbert class field.

(ii) Find the Hilbert class field of $K = \mathbb{Q}(\sqrt{229})$. Show that if $p \neq 229$ is a rational prime, then $p = x^2 + xy - 57y^2$ for some $x, y \in \mathbb{Z}$ if and only if $(p/229) = 1$ and $X^3 - 4X - 1 \equiv 0 \pmod{p}$ is soluble.

[The discriminant of $x^3 + px + q$ is $-4p^3 - 27q^2$. It may help to factor $(13 + \sqrt{229})$ and $(16 + \sqrt{229})$ in I_K.]