UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 28 May, 2004 9 to 12

PAPER 23

CATEGORY THEORY

Attempt **ONE** *question from Section A and* **TWO** *from Section B.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Section A

1 State and prove some form of the Adjoint Functor Theorem. Use it to establish the following result: 'If $\mathcal{C}$ and $\mathcal{D}$ are small categories and $F\colon \mathcal{C} \to \mathcal{D}$ a functor, then the functor $F^*\colon [\mathcal{D}, \mathbf{Set}] \to [\mathcal{C}, \mathbf{Set}]$ which sends G to GF has a left adjoint'.

2 P. Freyd has suggested that 'the purpose of category theory is to show that which is trivial is trivially trivial'. Write a short essay arguing **either** for **or** against this point of view, with reference to any of the major results proved in the course.

Section B

3 A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be *final* if, for each object B of $\mathcal{D}$, the arrow category $(B \downarrow F)$ is (nonempty and) connected. F is said to be a *discrete fibration* if, given $A \in \mathrm{ob}\,\mathcal{C}$ and $f\colon B \to FA$ in $\mathcal{D}$, there exists a unique $\tilde{f}\colon \tilde{B} \to A$ in $\mathcal{C}$ with $F\tilde{f} = f$.

(i) Given a commutative square

$$\begin{tikzcd} \mathcal{A} \arrow[r, "H"] \arrow[d, "F"'] & \mathcal{C} \arrow[d, "G"] \\ \mathcal{B} \arrow[r, "K"] & \mathcal{D} \end{tikzcd}$$

where F is final and G is a discrete fibration, show that there is a unique functor $L\colon \mathcal{B} \to \mathcal{C}$ with $LF = H$ and $GL = K$.

(ii) Show that any functor $F : \mathcal{C} \to \mathcal{D}$ can be factored as a final functor followed by a discrete fibration. [Hint: construct a category $\mathcal{E}$ whose objects are all connected components of the categories $(B \downarrow F)$, $B \in \mathrm{ob}\,\mathcal{D}$.]

(iii) Deduce from (i) that the factorization in (ii) is unique up to canonical isomorphism.

4 Explain what is meant by the statement that an adjunction is *monadic*, and by the *monadic length* of an arbitrary adjunction. State the Precise Monadicity Theorem.

Let $\mathcal{C}_n$ denote the category whose objects are sets A equipped with n partial unary operations $\alpha_i : A \rightharpoondown A$ $(1 \leqslant i \leqslant n)$, such that $\alpha_1(a)$ is defined for all $a \in A$ and, for $i > 1$, $\alpha_i(a)$ is defined iff ($\alpha_{i-1}(a)$ is defined and) $\alpha_{i-1}(a) = a$. Morphisms $A \to B$ in $\mathcal{C}_n$ are functions f such that $\alpha_i(f(a)) = f(\alpha_i(a))$ whenever $\alpha_i(a)$ is defined. Show that the forgetful functor $\mathcal{C}_n \to \mathcal{C}_{n-1}$ (which 'forgets' the operation α_n) has a left adjoint, and that the adjunction is monadic. Show also that the composite adjunction between $\mathcal{C}_n$ and $\mathcal{C}_0 = \mathbf{Set}$ has monadic length n. [Hint: show that the monad on $\mathcal{C}_m$ induced by the forgetful functor $\mathcal{C}_n \to \mathcal{C}_m$ and its left adjoint, for any $n > m$, is independent of n.]

5 Explain what is meant by a *filtered category*, and sketch the proof that filtered colimits commute with finite limits in **Set**.

Let $\mathcal{C}$ be a small category with finite limits and $F : \mathcal{C} \to \mathbf{Set}$ a functor. Show that the following conditions are equivalent:

(i) F preserves finite limits.

(ii) For any set A, the category $(A \downarrow F)$ has finite limits.

(iii) $(1 \downarrow F)^{\mathrm{op}}$ is filtered, where 1 denotes a one-element set.

(iv) F is expressible as a filtered colimit of representable functors.

6 Explain what is meant by a *regular category*, and by the statement that a regular category is *capital*. Sketch the proof that any small regular category $\mathcal{C}$ admits an isomorphism-reflecting regular functor to a capital regular category. Under what conditions does $\mathcal{C}$ admit an isomorphism-reflecting regular functor to **Set**? [Detailed proofs are not required.]