UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 3 June, 2004 9 to 12

PAPER 22

INTERSECTION COHOMOLOGY

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let X be a stratified pseudomanifold. Carefully define the terms *lower middle perversity* and *allowable singular simplex* (with respect to the lower middle perversity). Define the subcomplex $IS_*(X) \subset S_*(X)$ of *singular intersection chains.* Now suppose

$$X = \mathrm{Con}(Y) = \frac{Y \times [0,1)}{Y \times \{0\}}$$

is the cone on a stratified pseudomanifold Y with $\dim Y = 2n-1$. Prove that

$$IH_i(X) \cong \begin{cases} IH_i(Y \times (0,1)) & i < n \\ 0 & i \geqslant n. \end{cases}$$

Compute the intersection homology of the suspension

$$\mathrm{Susp}(Y) = \frac{Y \times [-1,1]}{Y \times \{-1,1\}}$$

in terms of $IH_*(Y)$. Briefly comment on your result paying particular attention to the case when Y is a Witt space. [*You may use standard results for computing intersection homology provided you state them carefully.*]

2 Let $(M, \partial M)$ be an oriented compact $(2m+1)$-dimensional topological manifold with collared boundary i.e. there is a neighbourhood U of ∂M in M which is homeomorphic (relative to the boundary) to $\partial M \times [0,1)$. Define

$$\widehat{M} = M \cup_{\partial M} \overline{\mathrm{Con}}(\partial M)$$

where the closed cone $\overline{\mathrm{Con}}(\partial M)$ is glued on to $(M, \partial M)$ via the natural identification of its boundary with ∂M. Let $\widehat{U} \subset \widehat{M}$ be the open set corresponding to $U \cup_{\partial M = \partial U} \overline{\mathrm{Con}}(\partial M)$. Show that there is a commutative diagram

$$\begin{array}{ccccccccccc}
\ldots & \longrightarrow & IH^i(\widehat{M}, \widehat{U}) & \longrightarrow & IH^i(\widehat{M}) & \longrightarrow & IH^i(\widehat{U}) & \longrightarrow & IH^{i+1}(\widehat{M}, \widehat{U}) & \longrightarrow & \ldots \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\ldots & \longrightarrow & H^i(M, U) & \longrightarrow & H^i(M) & \longrightarrow & H^i(U) & \longrightarrow & H^{i+1}(M, U) & \longrightarrow & \ldots
\end{array}$$

whose rows are long exact sequences.

State a necessary and sufficient condition for $\widehat{M}$ to be a Witt space. Under the assumption that this condition holds, deduce that

$$IH^i(\widehat{M}) \cong \begin{cases} H^i(M) & i \leqslant m \\ H^i(M, \partial M) & i > m. \end{cases}$$

How does this result change if we remove the condition that $\widehat{M}$ is a Witt space? [*You may use standard results for computing intersection cohomology provided you state them carefully.*]

Give an example of a manifold $(M, \partial M)$ with boundary of dimension $\geqslant 2$ such that the intersection cohomology (which you should explicitly compute) of the stratified pseudomanifold $\widehat{M}$ does **not** satisfy Poincaré duality.

3 [*All the spaces in this question are complex projective varieties. We consider them with the analytic, rather than the Zariski, topology. You may assume that they are compact oriented Witt spaces and, when nonsingular, that they are oriented manifolds. All the maps are projective maps of projective varieties. You may assume that they are proper, continuous and stratifiable.*]

Let $Gr(m,n)$ be the Grassmannian of complex m-dimensional linear subspaces of $\mathbf{C}^n$, in particular $Gr(1,n) \cong \mathbf{CP}^{n-1}$. Fix a point $W \in Gr(2,4)$ and define

$$S_W = \{V \in Gr(2,4) : \dim V \cap W \geqslant 1\}$$

and

$$F_W = \{(U,V) \in Gr(1,4) \times Gr(2,4) : U \leqslant V \cap W\}.$$

You may assume that S_W is singular with a natural stratification by $W \in S_W$ and $S_W \setminus \{W\}$ and that F_W is nonsingular.

Let $f : F_W \to S_W$ be the map $(U,V) \mapsto V$. Let $\mathcal{O}$ be the constant sheaf on F_W with stalk $\mathbf{Q}$. Show that $Rf_*\mathcal{O} \in D(S_W)$ is constructible with respect to the given stratification of S_W and that it satisfies the normalisation axiom. Identify the fibre $f^{-1}(W)$ and deduce that $Rf_*\mathcal{O}$ satisfies the support axiom.

Show that $\Sigma^{-6} D\,(Rf_*\mathcal{O}) \cong Rf_*\mathcal{O}$ where $D : D_S(S_W) \to D_S(S_W)$ is the Verdier dual functor. Deduce that $Rf_*\mathcal{O}$ satisfies the cosupport axiom. Hence show that $IH^*(S_W) \cong H^*(F_W)$.

By considering the map $g : F_W \to Gr(1,4) : (U,V) \mapsto U$, or otherwise, deduce that

$$IH^*(S_W) \cong H^*(\mathbf{CP}^1) \otimes H^*(\mathbf{CP}^2).$$

[*You may assume that if $\phi : E \to B$ is a projective map of nonsingular complex projective varieties which is topologically a fibration (with fibre F also a complex projective variety) and B is simply connected then $H^*(E) \cong H^*(B) \otimes H^*(F)$.*]

4 Carefully define the terms *pseudomanifold*, *Witt space* and *Witt space with collared boundary.* [*You need not define the term stratified space.*] Give an example of a stratified space which is not a pseudomanifold and of a pseudomanifold which is not a Witt space.

Let Ω_n^{Witt} be the bordism group of closed compact oriented n-dimensional Witt spaces. State a theorem which identifies $\Omega_{4k}^{\mathrm{Witt}}$ for $k \geqslant 1$, and give an example of a space whose class in Ω_4^{Witt} is non-zero.

Suppose X is a closed compact oriented $(2k+1)$-dimensional Witt space. Give a simple and explicit construction of a closed compact oriented $(2k+2)$-dimensional Witt space $(Y, \partial Y)$ with collared boundary such that $X \cong \partial Y$. Deduce that $\Omega_{2k+1}^{\mathrm{Witt}} \cong 0$.

Now suppose X is a closed compact oriented 2-dimensional Witt space. Fix a stratification and list the possible links of a 0-dimensional stratum. Show that there is a closed compact oriented topological 2-manifold $\widetilde{X}$ and a surjective continuous map $\pi : \widetilde{X} \to X$ whose fibres are 0-dimensional and which induces a homeomorphism $\pi^{-1}U \to U$ where $U \subset X$ is the open stratum. Hence, or otherwise, construct a compact oriented 3-dimensional Witt space $(Z, \partial Z)$ with collared boundary such that $X \cong \partial Z$. Deduce that $\Omega_2^{\mathrm{Witt}} \cong 0$.

Prove that a closed compact oriented 4-dimensional Witt space X which is a product of closed compact oriented Witt spaces of strictly positive dimension must be Witt null-bordant i.e. $[W] = 0 \in \Omega_4^{\mathrm{Witt}}$.

5 Let $CSh(X)$ be the category of bounded below cochain complexes of sheaves of rational vector spaces on a topological space X, with morphisms given by cochain maps. Define the *shift functor* $\Sigma : CSh(X) \to CSh(X)$. Let $\phi^* : \mathcal{E}^* \to \mathcal{F}^*$ be a morphism in $CSh(X)$. Define the *algebraic mapping cone* $C^*(\phi)$ of ϕ^* and show there are cochain maps

$$\mathcal{F}^* \to C^*(\phi) \to \Sigma\mathcal{E}^*.$$

Briefly explain how the *bounded below derived category* $D(X)$ is constructed from $CSh(X)$. Show that $\mathcal{E}^* \cong 0$ in $D(X)$ if, and only if, the cohomology sheaves $\mathcal{H}^i(\mathcal{E}^*) \cong 0$ for all $i \in \mathbf{Z}$. Explain what is meant by an *exact triangle* in $D(X)$. Suppose

$$0 \to \mathcal{E}^* \xrightarrow{\phi^*} \mathcal{F}^* \xrightarrow{\psi^*} \mathcal{G}^* \to 0$$

is a short exact sequence of complexes of sheaves in $CSh(X)$ i.e ϕ^* and ψ^* are cochain maps such that $0 \to \mathcal{E}^i \to \mathcal{F}^i \to \mathcal{G}^i \to 0$ is a short exact sequence of sheaves for each i. Show that there is a natural quasi-isomorphism $C^*(\phi) \to \mathcal{G}^*$. Deduce that there is an exact triangle

$$\mathcal{E}^* \to \mathcal{F}^* \to \mathcal{G}^* \to \Sigma\mathcal{E}^*$$

in $D(X)$.

Let $\jmath : S^1 \hookrightarrow S^2$ be the standard equatorial embedding and $\imath : S^2 \setminus S^1 \hookrightarrow S^2$ be the complementary open inclusion. By considering the exact triangle in $D(S^2)$ associated to the short exact sequence

$$0 \to \imath_! \imath^* \mathcal{O}_{S^2} \to \mathcal{O}_{S^2} \to \jmath_* \jmath^* \mathcal{O}_{S^2} \to 0$$

find a **non-zero** morphism μ in $D(S^2)$ such that the induced maps $\mathcal{H}^i(\mu) \cong 0$ between cohomology sheaves are 0 for all i.