UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 2 June, 2004 1.30 to 4.30

PAPER 21

GEOMETRIC INVARIANT THEORY

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 For an integer $n \geq 2$, consider the subgroup $\mathbb{Z}_n$ of $SL(2,\mathbb{C})$ generated by $\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix}$ where ζ denotes a primitive n^{th} root of unity.

Let $\mathbb{Z}_n \subset SL(2,\mathbb{C})$ act on the affine plane A^2 in the standard way. Find an embedding of the geometric quotient $A^2/\mathbb{Z}_n$ as a closed subvariety in some affine space. Give equations for this subvariety.

2 Let $G = SL(2,\mathbb{C})$, acting in the natural way on the vector space V of homogeneous degree 4 polynomials on $\mathbb{C}^2$. Find the set of stable points for the action of G on the projective space $P(V^*)$ of lines in V.

3 Let G be a complex reductive group, and let V be a complex representation of G. Show that the ring of invariants of G on the polynomial ring $\mathbb{C}[V]$ is finitely generated as a $\mathbb{C}$–algebra. You may use the properties of linear representations of G, including the Reynolds operator (the projection from a representation of G to its G-fixed subspace).

4 Show that for any rank-2 algebraic vector bundle E on a smooth compact complex algebraic curve, there is an upper bound on the degrees of all the line subbundles of E. Show that, for any such E, there is no lower bound on the degrees of line subbundles of E.