UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 7 June, 2004 9 to 11

PAPER 20

LIE GROUPS

Attempt **TWO** *questions.*

There are **three** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (i) Let G be a Lie group and LG its Lie algebra. Define the exponential map

$$\exp : LG \to G$$

and show that it is a homomorphism if and only if G is abelian. Deduce that a connected abelian Lie group is the product of a torus and a vector space, $G \cong T^r \times \mathbb{R}^s$.

(ii) Formulate the concept of a *complex analytic* Lie group. Show that if G is such a group which is compact and connected, then G is isomorphic to $\mathbb{C}^n/B$ where B is a discrete subgroup of $\mathbb{C}^n$.

(Hint: by using properties of complex analytic maps defined on a compact domain, show that the adjoint representation of G is trivial.)

2 (i) Let G be a compact topological group. Outline the main steps in the proof of the (Peter-Weyl) theorem, which states that every continuous function $f : G \to \mathbb{C}$ can be approximated by functions of the form $\mathrm{Trace}(\alpha\theta(g))$, where α is a $\mathbb{C}$-linear endomorphism of the finite dimensional vector space V, and $\theta : G \to \mathrm{Hom}_{\mathbb{C}}(V, V)$. Deduce that if f is a class function, f can be approximated by a linear combination of irreducible complex characters.

(ii) If the compact group G satisfies the descending chain condition for closed subgroups, show that G is a Lie group.

3 Write an essay on the relation between { Lie groups G and smooth homomorphisms} and {Lie algebras $\mathfrak{g}$ and $\mathbb{R}$-linear, bracket preserving maps}. At the very least your essay should include a definition of the Lie algebra LG and how the algebra homomorphism $L\phi : LG_1 \to LG_2$ induced by $\phi : G_1 \to G_2$ determines ϕ. However informally you should also consider the following points:

(i) How are the groups G_1 and G_2 related if $LG_1 \cong LG_2$?

(ii) Given a Lie algebra $\mathfrak{h}$ how might you construct a corresponding group H?

(iii) Given an algebra homomorphism $\theta : \mathfrak{g}_1 \to \mathfrak{g}_2$ how might you construct a group homomorphism (defined on all of G_1) $\phi : G_1 \to G_2$, such that $L\phi = \theta$?