UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 27 May, 2004 1.30 to 4.30

PAPER 2

TOPICS IN GROUP THEORY

Attempt **THREE** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let G be a finite group.

(a) Define the Frattini subgroup $\Phi(G)$ and Fitting subgroup $F(G)$ of G. Prove that $F(G)$ exists.

(b) Show that $\Phi(G) \leq F(G)$.

(c) If G is soluble prove that $F(G)$ contains its own centraliser.

(d) Let G be a finite group. The *socle* of G is the subgroup $\mathrm{Soc}(G)$ generated by all the minimal normal subgroups of G. Show that $F(G) \leq C_G\big(\mathrm{Soc}(G)\big)$.

(e) Prove that $F\big(G/\Phi(G)\big) = F(G)/\Phi(G)$.

2 Write an essay on soluble groups. You should include a treatment of Hall π-subgroups.

3 Let G be a doubly transitive permutation group on a finite set Ω of size n and let N be a minimal normal subgroup of G.

(a) If N is regular show that N is elementary abelian of order $n = p^d$ for some prime number p and G is a subgroup of $AGL(d,p)$.

(b) If N is imprimitive show that N is regular.

(c) If N is primitive and non-regular show that N is a non-abelian simple group and $N \leq G \leq \mathrm{Aut}(N)$.

(d) If G is 4-transitive on Ω and N is regular, show that $n = 4$ and $G \cong S_4$.

4 (a) Let G be a group acting primitively on a set Ω and suppose that G is generated by $\{A^g : g \in G\}$, where A is an abelian normal subgroup of G_α for some $\alpha \in \Omega$. If N is a normal subgroup of G, show that either $G' \leq N$ or $N \leq G_{[\Omega]}$ (where $G_{[\Omega]}$ is the pointwise stabiliser of Ω).

(b) Prove that the special linear group $SL(n,\mathbb{F})$ of dimension $n \geq 2$ over a field $\mathbb{F}$, is generated by transvections.

(c) Prove that $PSL(n,\mathbb{F})$ is simple if $n \geq 3$ or if $n = 2$ and $|\mathbb{F}| > 3$.

(d) Outline the modifications necessary to prove the corresponding result for symplectic groups.

5 (a) Describe how to associate a partition-valued function on the set of irreducible polynomials over $\mathbb{F}_q$ to each conjugacy class of $GL(n,q)$ and give an expression for the size of the corresponding centraliser.

(b) Show that the number of conjugacy classes in $GL(n,q)$ is the coefficient of t^n in

$$\prod_{i=1}^{\infty} \frac{1-t^i}{1-qt^i}\,.$$

(c) Show that the number of unipotent elements in $GL(n,q)$ is q^{n^2-n}.

[You may assume that $\displaystyle\prod_{i=1}^{\infty} \frac{1}{1-st^i} = \sum_{k=0}^{\infty} \frac{s^k t^k}{\phi_k(t)}$ where $\displaystyle\phi_k(t) = \prod_{i=1}^{k}(1-t^i)$.]

(d) Let $f(t)$ be a monic irreducible quadratic polynomial over $\mathbb{F}_q$. Show that the number of elements of $GL(2m,q)$ whose minimal polynomial is a power of $f(t)$ is

$$q^{2m^2-2m}|GL(2m,q)|/|GL(m,q^2)|.$$

6 (a) Define the symplectic group $Sp(2m,q)$ and show that

$$|Sp(2m,q)| = q^{m^2}\prod_{i=1}^{m}(q^{2i}-1).$$

(b) Let $\Omega = \{1,2,\ldots,n\}$ and let V be the set of all subsets of Ω. With addition defined by symmetric difference,

$$A+B = (A\cup B)\setminus(A\cap B),$$

V can be regarded as a vector space over the field $\mathbb{F}_2$ of order 2. Show that the map $(\ ,\ ) : V\times V \to \mathbb{F}_2$ defined by

$$(A,B) = |A\cap B| \pmod 2,$$

is a symmetric bilinear form on V, preserved by the natural action of the symmetric group S_n.

If n is even, show that $(\ ,\ )$ induces a non-singular alternating bilinear form on $<\Omega>^{\perp} / <\Omega>$.

(c) Prove that $Sp(4,2) \cong S_6$.