UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 4 June, 2004 9 to 12

PAPER 19

RIEMANNIAN GEOMETRY

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $\gamma : [0, \infty) \to M$ be a geodesic in a locally symmetric space (i.e. $\nabla R = 0$, where R is the curvature tensor of M) and let $(p, v) = (\gamma(0), \dot{\gamma}(0))$. Consider the linear map $K_v : T_pM \to T_pM$ given by

$$K_v(x) = R(v, x)v \quad x \in T_pM.$$

(a) Prove that K_v is self-adjoint.

(b) Choose an orthonormal basis $\{e_1, \ldots, e_n\}$ of T_pM which diagonalises K_v, i.e., $K_v(e_i) = \lambda_i\, e_i$, for $i = 1, \ldots, n$. Consider $e_i(t)$, the parallel transport of e_i along γ. Show that for all t, $K_{\dot{\gamma}(t)}(e_i(t)) = \lambda_i\, e_i(t)$, where λ_i is independent of t.

(c) Solve the Jacobi equation and show that the conjugate points to p along γ are given by $\gamma(\pi\, k/\sqrt{\lambda_i})$, where k is a positive integer and λ_i is a positive eigenvalue of K_v.

2 (a) What is a complete Riemannian manifold? State the Hopf-Rinow theorem.

(b) Let $f : M \to N$ be a local isometry between Riemannian manifolds. Show that if M is complete, then N is also complete. Is the converse true? Is the converse true if f is a covering map?

(c) Show that if M is complete, then f is a covering map.

3 (a) Let M be a complete Riemannian manifold. Explain what is meant by M being disconnected at infinity. Show that if M is disconnected at infinity, it must contain a line.

(b) State the Cheeger-Gromoll splitting theorem.

(c) Let M be a closed 3-manifold whose universal covering is not contractible. Show that $M \times \mathbf{R}$ does not admit a complete Ricci flat metric.

4 (a) Let M^n be a complete manifold with $\mathrm{Ric} \geqslant k > 0$. Show that M is compact and $\mathrm{diam}(M) \leqslant \pi/\sqrt{k}$. [*You may assume the formula for the second variation of energy.*]

(b) Is the result in (a) true if we just assume $\mathrm{Ric} > 0$? Does there exist a metric with positive Ricci curvature on $S^2 \times S^1$? Justify your answers.

(c) Let M be an orientable Riemannian manifold of even dimension and positive sectional curvature. Show that any closed geodesic γ in M is homotopic to a closed curve with length strictly smaller than that of γ. [*You may assume the formula for the second variation of energy.*]

5 (a) Let M^n be a complete Riemannian manifold with $\mathrm{Ric} \geqslant 0$. Let Γ be any finitely generated group of isometries acting properly discontinuously on M. Prove that Γ has polynomial growth of degree less than or equal to n. [*You may use a comparison result for Ricci curvature.*]

(b) Show that a complete Riemannian manifold M^n with $\mathrm{Ric} \geqslant 0$ such that

$$\lim_{r \to +\infty} V(p,r)/w_n r^n = 1$$

for some $p \in M$, must be isometric to the Euclidean space. ($V(p,r)$ is the volume of a ball of centre p and radius r in M and ω_n is the volume of the unit ball in the Euclidean n-space.) [*You may use a comparison result for Ricci curvature.*]

(c) By using (a) or otherwise exhibit a connected Lie group which does not admit a bi-invariant metric. [*You may assume that bi-invariant metrics on a Lie group have non-negative sectional curvature.*]