UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 2 June, 2004 9 to 12

PAPER 17

KNOT THEORY

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define the Kauffman bracket (Laurent) polynomial $\langle D\rangle$ of a diagram D of a classical link. Investigate the way that $\langle D\rangle$ changes when D is changed by a Reidemeister move and define the Jones polynomial of an oriented link.

If an oriented link L has a connected diagram D with n crossings, show that the breadth of its Jones polynomial is at most n and is equal to n if D is a reduced alternating diagram.

If K_1 and K_2 are oriented alternating knots show that

$$c(K_1 + K_2) = c(K_1) + c(K_2)$$

where $c(K)$ is the minimal number of crossings in any diagram of the knot K.

Suppose that D is a reduced alternating diagram of a two-component link L. Show that if L is a split link then D is a split diagram (that is, if L can be separated by a 2-sphere embedded in $S^3 - L$ then D can be separated by a simple closed curve in $S^2 - D$).

2 Explain what is meant by the *sum* of two oriented knots and what is meant by saying that a knot is *prime*. Prove that any knot can be expressed as a sum of prime knots.

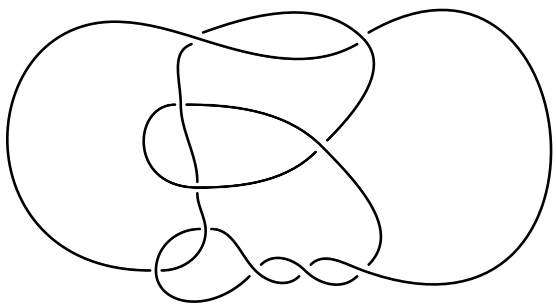
For every pair of coprime integers p and q, define the (p, q)-torus-knot and prove that it is a prime knot provided that $|p| \geqslant 2$ and $|q| \geqslant 2$.

3 What is a Seifert matrix for a knot K in S^3? By considering the infinite cyclic cover of the complement of K, define the Alexander module and the Alexander polynomial $\Delta_K(t)$ of K. Explain, with detailed proof, how $\Delta_K(t)$ can be calculated from any Seifert matrix for K. [General results about covering spaces, and results about finitely generated modules, may be assumed without proof.]

If now K is the knot shown below, prove that, up to multiplication by a unit,

$$\Delta_K(t) = 5t^4 - 15t^3 + 21t^2 - 15t + 5.$$

What is the genus of K? Is K a prime knot?



4 Explain without proof how to write down a presentation of the group of a knot, namely the fundamental group of the complement of the knot, based on a planar diagram of the knot. Show that the group of a trefoil knot T (the knot 3_1) has a presentation with two generators and one relator.

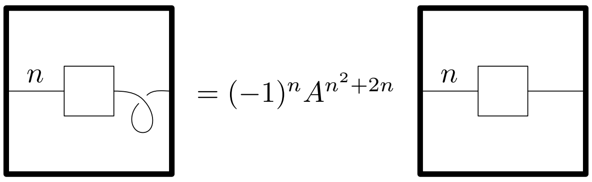
Suppose that a knot group G has a presentation with generators $\{x_1, x_2, \ldots, x_n\}$ and relators $\{r_1, r_2, \ldots, r_m\}$. Describe in detail the Fox free differential calculus that associates, to the group presentation, a matrix $\left(\frac{\partial r_i}{\partial x_j}\right)$ and explain, giving full proofs, how this matrix is related to the Alexander module of the knot.

Determine the Alexander polynomial of T. Prove that the knot $T + T$ cannot have a presentation of its knot group consisting of just two generators and one relator.

[General results about covering spaces, and results about finitely generated modules, may be assumed without proof.]

5 Explain the idea of *skein theory* associated with oriented 3-manifolds with boundary. Define the Temperley-Lieb algebras, with parameter a fixed complex number A, and develop the theory of the Jones-Wenzl idempotents in these algebras.

Prove the identity in the nth Temperley-Lieb algebra depicted below where, using the usual conventions, the small square represents the Jones-Wenzl idempotent.



Explain with proofs how, for suitable choices of A, this skein theory can be used to give, for closed oriented 3-manifolds, well defined 'quantum' invariants that are associated to the theory of the Jones polynomial. Illustrate this by finding an expression in terms of A for such invariants for real projective 3-space (which is the 3-manifold obtained from S^3 by surgery on the unknot with framing 2).

[General results concerning the obtaining of 3-manifolds by surgery on links may be quoted without proof.]