UNIVERSITY OF
CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 1 June, 2004 1.30 to 4.30

PAPER 16

ALGEBRAIC TOPOLOGY

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

All (co)homology groups have $\mathbb{Z}$ coefficients. You may assume (i) manifolds are homotopy equivalent to cell complexes and (ii) manifolds have finite type, i.e. admit a cover by finitely many open discs such that all iterated intersections are empty or homeomorphic to open discs. Standard algebraic results such as the 5-lemma and existence of long exact sequences may be used throughout. Results on homotopy lifting may be used without proof if carefully stated. You may also assume computations of $\pi_*(S^n)$.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 What is a chain complex and what are the homology groups of a chain complex?

Define the singular chain complex $C_*(X)$ for a topological space X, and show the singular homology $H_*(X)$ depends only on the homotopy type of X.

Quoting any other results you use, compute $H_*(\Sigma_3)$ where Σ_3 is a closed surface of genus 3. Draw on Σ_3 simple closed curves C_1 and C_2 for which there is no homotopy equivalence $f : \Sigma_3 \to \Sigma_3$ with $f(C_1) = C_2$ but for which there does exist a map $g : \Sigma_3 \to \Sigma_3$ with $g(C_1) = C_2$. (Justify your answers briefly.)

2 What is a cell complex? Define the cellular boundary operator and cellular homology; you may assume standard facts about the (relative) singular homology groups of cell complexes and their subcomplexes.

Hence compute $H_*^{\text{cell}}(S^k)$ and $H_*^{\text{cell}}(\mathbb{C}P^n)$, and also $H_*^{\text{cell}}(S^{2n-2} \times S^3)$ for $n > 3$.

By collapsing the 2n-skeleton to a point, or otherwise, show there is a map $\phi : S^{2n-2} \times S^3 \to \mathbb{C}P^n$ which induces $\phi_* = 0$ on homotopy groups π_* and reduced homology $\tilde{H}_*$, but such that ϕ is <u>not</u> homotopic to a constant map. [Hint: for the very last part use homotopy lifting.]

3 What does it mean to say a (topological) manifold M is oriented?

Let M be a connected manifold of dimension n. Prove $H^n_{ct}(M)$ is cyclic, and isomorphic to $\mathbb{Z}$ if M is oriented.

Give a careful statement of the Poincaré duality theorem.

Which of the following sequences of abelian groups $(G_i | 0 \leqslant i \leqslant 4)$ can arise as the homology groups $H_i(M, \mathbb{Z})$ of a closed oriented 4-manifold? For each of the cases (a), (b), (c) give a short justification for your answer.

(a) $G_0 = \mathbb{Z} \quad G_1 = 0 \quad G_2 = \mathbb{Z} \quad G_3 = \mathbb{Z} \oplus \mathbb{Z} \quad G_4 = \mathbb{Z}$,

(b) $G_0 = \mathbb{Z} \quad G_1 = \mathbb{Z} \oplus \mathbb{Z} \quad G_2 = 0 \quad G_3 = \mathbb{Z} \oplus \mathbb{Z} \quad G_4 = \mathbb{Z}$,

(c) $G_0 = \mathbb{Z} \quad G_1 = 0 \quad G_2 = \mathbb{Z} \quad G_3 = \mathbb{Z}/2\mathbb{Z} \quad G_4 = \mathbb{Z}$.

4 State the Thom isomorphism theorem for an oriented real vector bundle $E \to X$ and define the Euler class $e(E)$ of E.

If X is a smooth closed manifold and $Y \subseteq X$ is a smooth compact cooriented submanifold, explain how to associate a cohomology class $\varepsilon_Y \in H^d(X)$ to Y, where $d = \text{codim}(Y \subseteq X)$. By computing ε_Δ, where $\Delta \subseteq X \times X$ is the diagonal submanifold, prove the Lefschetz fixed point theorem:- if $f : X \to X$ has $L(f) \neq 0$, where $L(f) = \sum_{i \geqslant 0} (-1)^i \text{ tr } (f^* : H^i(X) \to H^i(X))$, then f has a fixed point.

Quoting any other results you wish, deduce

(i) if $f : S^n \to S^n$ has degree 0 then $\exists x, y \in S^n$ such that

$$f(x) = x, \ f(y) = -y$$

(ii) $\mathbb{C}P^2$ is not the total space of any covering map of degree $d \geqslant 2$.

[You may assume the formula $\int_X \varepsilon_Y \smile \alpha = \int_Y \alpha|_Y$.]

5 What is the "tautological" vector bundle $E \to Gr_k(\mathbb{C}^n)$ over the Grassmannian of complex k-planes in $\mathbb{C}^n$?

Assuming any existence results for inner products, show the set of rank k complex vector bundles over a compact space X up to isomorphism, $\text{Vect}^k_{\mathbb{C}}(X)$, is in bijective correspondence with the set $[X, Gr_k\mathbb{C}^\infty]$ of homotopy classes of maps from X to the Grassmannian of k-planes in $\mathbb{C}^\infty$.

Using the existence of a fibre bundle $U_k \to \nu \to Gr_k\mathbb{C}^\infty$ with contractible total space, show $\pi_q(Gr_k\mathbb{C}^\infty) \cong \pi_{q-1}(U_k)$ for $q \geqslant 1$.

Finally, since $SU_2 \cong S^3$, deduce $\text{Vect}^2_{\mathbb{C}}(S^4) \cong \mathbb{Z}$.

[You may assume that if Z is a simply connected space, $\pi_q(Z) = [S^q, Z]$ for $q \geqslant 2$.]