UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 27 May, 2004 9 to 12

# PAPER 15

# ALGEBRAIC GEOMETRY

*Attempt* **THREE** *questions.*

*There are* **four** *questions in total.*

*The questions carry equal weight.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Suppose that $\Lambda$ is a lattice in the complex plane $\mathbb{C}$. Describe briefly the construction and basic properties of the Weierstrass $\wp$ and $\sigma$ functions associated to $\Lambda$. Show that if $P_1, \ldots, P_n, Q_1, \ldots, Q_n \in \mathbb{C}$, then there is a meromorphic function on $\mathbb{C}/\Lambda$ whose divisor of zeros is the divisor $\sum [P_i]$ and whose divisor of poles is the divisor $\sum [Q_i]$ if and only if $\sum P_i \equiv \sum Q_i$ modulo $\Lambda$.

**2** (i) State and prove the theorem of the cube for an elliptic curve over $\mathbb{C}$. What modification of the statement is necessary for an elliptic curve over a number field?

(ii) Explain briefly how $\wp$ and its derivative $\wp'$ can be used to embed $\mathbb{C}/\Lambda$ as a smooth cubic curve $C$ in $\mathbb{P}^2_{\mathbb{C}}$. Show that points $u, v, w \in \mathbb{C}$ have colinear images in $C$ if and only if $u + v + w \in \Lambda$. Deduce that the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ \wp(u) & \wp(v) & \wp(w) \\ \wp'(u) & \wp'(v) & \wp'(w) \end{vmatrix}$$

is zero if and only if $u + v + w \in \Lambda$.

**3** Explain the construction of height functions for projective varieties over number fields, and give a proof of Northcott's lemma.

**4** Write an essay on normalized heights on an elliptic curve over a number field, including a description of their relevance to a proof of the Mordell-Weil theorem.