UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 1 June, 2004 9 to 12

PAPER 14

DIFFERENTIAL GEOMETRY

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

All manifolds and related concepts should be assumed to be smooth.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Define smooth vector fields on a manifold. Show that the tangent bundle TM over an n-dimensional manifold M is isomorphic to the product vector bundle $M \times \mathbb{R}^n$ if and only if M admits a collection of n smooth vector fields $X_1, \ldots, X_n$, such that $X_1(p), \ldots, X_n(p)$ is a basis of the tangent space T_pM, for any $p \in M$.

Explain what is meant by a left-invariant vector field on a Lie group G and show that any left-invariant vector field is smooth. Show that the tangent bundle TG is isomorphic to the product vector bundle $G \times \mathbb{R}^{\dim G}$.

[*Basic properties of the differential of a smooth map between manifolds can be used without proof provided these are clearly stated.*]

2 Define a smooth free right action of a Lie group G on a manifold P. Give a definition of a principal G-bundle $\pi : P \to B$ over a manifold B. Show that if Φ_1, Φ_2 are two local trivializations of π over some (overlapping) neighbourhoods N_1, N_2 in B then $\Phi_2 \circ \Phi_1^{-1}(x, g) = (x, \psi(x)g)$, for some smooth map $\psi : N_1 \cap N_2 \to G$, where $x \in B$, $g \in G$.

Let the map $p : \mathbb{R}P^3 \to \mathbb{C}P^1$ be given by $p(x_0 \!:\! x_1 \!:\! x_2 \!:\! x_3) = (x_0 + ix_1) : (x_2 + ix_3)$. Identify a smooth free right action of $U(1)$ on $\mathbb{R}P^3$ whose orbits are precisely the fibres of p over points in $\mathbb{C}P^1$. Show that p is a principal $U(1)$-bundle by constructing appropriate local trivializations over the neighbourhoods $\{z : 1 \mid z \in \mathbb{C}\}$ and $\{1 : z \mid z \in \mathbb{C}\}$ in $\mathbb{C}P^1$. Calculate the transition function between these local trivializations.

3 Let A be a connection on a vector bundle E. Using local coordinates on the base manifold and a local trivialization of E, give an explicit local formula for the covariant derivative d_A induced by A and acting on the sections of E. Explain how to extend d_A, using an appropriate version of the Leibniz rule, to the differential forms with values in E and to the differential forms with values in the endomorphism bundle $\operatorname{End} E$. For both cases, include explicit formulae for d_A in local trivializations.

Define the curvature $F(A)$ of a connection A, showing that $F(A)$ is a well-defined 2-form with values in $\operatorname{End} E$. Prove the Bianchi identity $d_A F(A) = 0$.

By using the Bianchi identity or otherwise, show that if E is a vector bundle of rank 1 then $F(A)$ is a closed form and its de Rham cohomology class is independent of the choice of connection A.

[*Preliminary results on connections may be used without proof provided these are clearly stated.*]

4 Explain what is meant by a geodesic on a Riemannian manifold and deduce a system of ordinary differential equation satisfied by geodesics in local coordinates.

Given a geodesic $\gamma(t)$, defined for $|t| < \varepsilon$ ($\varepsilon > 0$), show that for some $\delta > 0$ the family of velocity vectors $\dot{\gamma}(t)$, $|t| < \delta$, can be extended to a smooth vector field defined on a neighbourhood of $\gamma(0)$. Prove that the velocity vectors $\dot{\gamma}(t)$ have constant length.

Find all geodesics on the sphere S^n with the metric induced by the standard embedding in Euclidean $\mathbb{R}^{n+1}$.

5 Define the Levi–Civita connection on a Riemannian manifold and prove that every Riemannian manifold has a unique Levi–Civita connection.

Define the curvature of a Riemannian manifold and state the symmetry relations satisfied by the curvature components. Define the Ricci curvature and show that it is a symmetric bilinear form on the tangent spaces. Show that on a Riemannian manifold of dimension *three* the value of the (full) curvature at any point is determined by the Ricci curvature at that point.