UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 1 June, 2004 9 to 11

PAPER 12

RAMSEY THEORY

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let m be a positive integer. Using van der Waerden's theorem, show that there exists an ultrafilter on $\mathbb{N}$, each member of which contains an m-term arithmetic progression.

[Hint: Which are the sets that *must* belong to such an ultrafilter?]

Show similarly that there exists an ultrafilter on $\mathbb{N}$, each member of which contains arbitrarily long arithmetic progressions.

Does there exist an ultrafilter on $\mathbb{N}$, each member of which contains an infinite arithmetic progression?

2 State and prove the Hales-Jewett theorem, and deduce van der Waerden's theorem. Prove the strengthened van der Waerden's theorem.

State Gallai's theorem, and show how to deduce it from the Hales-Jewett theorem.

3 State and prove Rado's theorem.

[You may assume that, for any m, p, c, whenever $\mathbb{N}$ is finitely coloured there is a monochromatic (m, p, c)-set.]

Deduce that, for any k, whenever $\mathbb{N}$ is finitely coloured there exist $x_1, x_2, \ldots, x_k$ with $FS(x_1, x_2, \ldots, x_k)$ monochromatic.

4 What does it mean to say that a subset of $\mathbb{N}^{(\omega)}$ is *Ramsey*? Give an example of a set that is not Ramsey. Prove that every τ-open set is Ramsey.

What does it mean to say that a subset of $\mathbb{N}^{(\omega)}$ is *completely Ramsey*? Give an example of a set that is Ramsey but not completely Ramsey.

Give an example of a (non-empty) $*$-open set that is τ-nowhere-dense. By choosing a suitable subset of this set, show that it is *not* the case that every τ-nowhere-dense set is completely Ramsey.