UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 1 June, 2004 1.30 to 4.30

PAPER 11

PROBABILISTIC COMBINATORICS

Attempt **FOUR** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let A_i, $1 \leq i \leq n$ be events and let $J_i \subset [n]$ be such that A_i is independent of the system $\{A_j : j \neq i, j \notin J_i\}$. Let $p_i = \Pr A_i$ and let $\Delta = \frac{1}{2}\sum_i \sum_{j \in J_i} \Pr(A \cap B)$, as usual. Let W be the number of A_i that occur and let $\lambda = \mathbb{E}W = \sum_i p_i$. Prove the Stein-Chen bound:

$$\mathrm{d}_{TV}(\mathcal{L}(W)\,,\,\mathrm{Po}(\lambda)) \leq \min(1, \lambda^{-1}) \left[\sum_i^n p_i^2 + \sum_i p_i \sum_{j \in J_i} p_j + 2\Delta \right].$$

[You may assume without proof the standard inequality $\Delta g \leq \min(1, \lambda^{-1})$.]

A random 3-uniform hypergraph of order n is constructed by choosing the edges independently and at random with probability p from the $\binom{n}{3}$ possible edges. Let W be the number of complete subgraphs of order 4 and let $\lambda = \mathbb{E}W = \binom{n}{4}p^4$. Show that

$$\mathrm{d}_{TV}(\mathcal{L}(W)\,,\,\mathrm{Po}(\lambda)) \leq 4np^4 + 4np^3.$$

2 State and prove the Local Lemma.

Deduce the Ramsey bound $R(k) \geq (\sqrt{2}/e + o(i))\, k\, 2^{k/2}$.

The vertices of a cycle of length 12n are partitioned into n sets each of size 12. Show that it is possible to select an independent set of n vertices, one from each set.

3 Let $\Omega = \prod_{i=1}^n A_i$ be a product probability space and let $f : \Omega \to \mathbb{R}$ be such that $|f(\omega) - f(\omega')| \leq c_i$ whenever ω and ω' differ in only the i'th co-ordinate. Prove that $\Pr\{|f - \mathbb{E}f| \geq t\} \leq 2\exp(-2t^2 \,/\, \sum_i c_i^2)$.

Let $A \subset \mathcal{P}[n]$ and let $A_t = \{y \in [n] : \exists x \in A, d(x,y) \leq t\}$ where $d(x,y)$ is the Hamming distance. Prove that, if $|A| \geq \epsilon 2^n$ and $t = \sqrt{2n \log(2/\epsilon)}$, then $|A_t| \geq (1-\epsilon)2^n$.

4 Describe how the semi-random method can be used to prove that, for all $r \in \mathbb{N}$ and $\epsilon > 0$, there exists $\delta > 0$ so that, for any r-uniform hypergraph G whose degrees lie between $(1-\epsilon)D$ and $(1+\epsilon)D$ and whose co-degrees are bounded by δD, the chromatic index $\chi'(G)$ of G satisfies $\chi'(G) \leq (1+\epsilon)D$.

Your essay should make clear the main steps of the proof, and should show how the standard probabilistic tools are applied.

5 Let $\Omega = \prod_{i=1}^n X_i$ be a product of finite probability spaces. Define the Talagrand distance function $d_{\mathrm{T}}(x, A)$, where $x \in \Omega$ and $A \subset \Omega$.

Prove Talagrand's inequality $\mathbb{E}\exp\{d_{\mathrm{T}}^2(x,A)/4\} \leq 1/\Pr A$.

Explain how it can be used to prove that the length of a longest increasing subsequence of a random sequence is concentrated near its mean value.

6 Let $X = (X_1, \ldots, X_n)$ be a sequence of random variables and, for $A \subset [n]$, let $X_A = (X_i : i \in A)$. Let $\mathcal{A} \subset \mathcal{P}[n]$ be such that $|\{A \in \mathcal{A} : j \in A\}| \geq k$ for all $j \in [n]$. Prove that $kH(X) \leq \sum_{A \in \mathcal{A}} H(X_A)$, where H is the entropy function. [Standard facts about entropy should be stated if used, but need not be proved.]

A bipartite graph is (a, b)-*regular* if the vertices in one class have degree a and those in the other class have degree b. Show that an (a, b)-regular bipartite graph of order n has at most $(2^a + 2^b - 1)^{n/(a+b)}$ independent subsets.