UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 3 June, 2004 1.30 to 4.30

# PAPER 10

# BANACH ALGEBRAS

*Attempt* **THREE** *questions.*

*There are* **five** *questions in total.*

*The questions carry equal weight.*

*All Banach algebras should be taken to be over the complex field, and to be non-zero. For an element $x$ of a Banach algebra, $r(x)$ denotes the spectral radius of $x$.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Let $A$ be a Banach algebra with identity element 1 and given norm $\|\,.\,\|$. Let $S$ be a bounded subset of $A$ such that $xy \in S$ whenever $x \in S$ and $y \in S$. Prove that there is a unital algebra-norm $\|\,.\,\|_1$ on $A$, that is equivalent to $\|\,.\,\|$ and is such that $\|s\|_1 \leqslant 1$ for all $s \in S$.

Define the *spectrum* $\operatorname{Sp} x$ of an arbitrary element $x \in A$ and state, *without proof* a formula that gives the spectral radius $r(x)$ in terms of the norm. (N.B. the spectral radius is here *defined* as $r(x) = \sup\{|\lambda| : \lambda \in \operatorname{Sp} x\}$.) Prove that if $r(x) < 1$ then $\{x^n : n \geqslant 1\}$ is a bounded subset of $A$.

Let $F = \{x_1, \ldots, x_n\}$ be a finite subset of $A$ such that $x_i x_j = x_j x_i$ for all $i, j = 1, \ldots, n$ and let $\epsilon > 0$. By using the result of the last paragraph, show that there is a unital algebra-norm $\|\,.\,\|_0$ on $A$, equivalent to $\|\,.\,\|$ and such that $\|x_j\|_0 < r(x_j) + \epsilon$ $(j = 1, \ldots, n)$.

Deduce (or prove otherwise), that if $a, b \in A$ satisfy $ab = ba$, then

$$r(ab) \leqslant r(a)r(b) \quad \text{and} \quad r(a+b) \leqslant r(a) + r(b)\,.$$

**2** Let $A$ be a Banach algebra with identity, let $x \in A$ and let $U$ be an open neighbourhood of $\operatorname{Sp} x$ in $\mathbb{C}$. Prove that there is a unique continuous, unital homomorphism $\Theta_x : \mathcal{O}(U) \to A$ such that $\Theta_x(Z) = x$ (where $Z$ is the function $Z(\lambda) = \lambda$ $(\lambda \in U)$).

Prove also that, for every $f \in \mathcal{O}(U)$, $\operatorname{Sp} \Theta_x(f) = f(\operatorname{Sp} x)$.
[*Any form of the Runge approximation theorem may be quoted without proof.*]

Now suppose that $(x_n)_{n \geqslant 1}$ is a sequence in $A$ with $x_n \to x$ as $n \to \infty$. Prove that:

(a) $\operatorname{Sp} x_n \subset U$ for all sufficiently large $n$;

(b) for every $f \in \mathcal{O}(U)$, $\Theta_{x_n}(f) \to \Theta_x(f)$ as $n \to \infty$.

Suppose, in addition, that $U = V \cup W$, where $V, W$ are open sets with $V \cap W = \varnothing$ and with $\operatorname{Sp} x \cap V \neq \varnothing$. Prove that:

(c) $\operatorname{Sp} x_n \cap V \neq \varnothing$ for all sufficiently large $n$.

**3** Let $A$ be a Banach algebra with identity, with $A$ not necessarily commutative. Define a *primitive ideal* of $A$. Show that every maximal two-sided ideal of $A$ is primitive.

Define the (Jacobson) radical $J = J(A)$ of $A$ to be the intersection of all the primitive ideals of $A$. Prove that:

(i) $J$ is the intersection of all the maximal left ideals of $A$;

(ii) $J = \{x \in A : 1 + yx \text{ is invertible for every } y \in A\}$;

(iii) $J$ is the greatest ideal of $A$ that is included in $N(A) \equiv \{x \in A : r(x) = 0\}$.

What does it mean to say that $A$ is *semisimple*?

Let $X$ be a Banach space, of dimension greater than 1, and let $B = B(X)$ be the algebra of all bounded linear operators on $X$. Prove that $B$ is semisimple, but that $N(B) \neq \{0\}$. [*Hint*: let $f$ be a non-zero element of $X^*$ and take some non-zero $x_0 \in X$ with $f(x_0) = 0$. Define $T \in B(X)$ by $T(x) = f(x)x_0$.]

Show that $\{0\}$ is a primitive ideal of $B$, but that, provided $X$ is infinite-dimensional, $\{0\}$ is not a maximal ideal.

**4** (i) Let $(f_n)_{n\geqslant 1}$ be a decreasing sequence of non-negative, continuous real-valued functions on a compact Hausdorff space $K$, and define $f(x) = \lim_{n\to\infty} f_n(x)$ $(x \in K)$. Prove that $\sup_K f_n \to \sup_K f$ as $n \to \infty$.

(ii) Let $A$ be a Banach algebra with identity and let $f$ be a holomorphic $A$-valued function on $\mathbb{C}$. Prove that for every $R > 1$,

$$r\big(f(1)\big)^2 \leqslant \sup_{|z|=R} r\big(f(z)\big) \sup_{|z|=R^{-1}} r\big(f(z)\big)\,.$$

Deduce that if the function $z \mapsto r\big(f(z)\big)$ is bounded on $\mathbb{C}$ then it is constant.

**5** Let $A$ be a $C^*$-algebra (with identity); a linear functional $f$ on $A$ is said to be *positive* if $f(x^*x) \geqslant 0$ for every $x \in A$. Let $f$ be a positive linear functional on $A$; prove that:

(i) $f(x^*) = \overline{f(x)}$ for all $x \in A$;

(ii) $|f(x^*y)|^2 \leqslant f(xx^*)f(yy^*)$ for all $x, y \in A$;

(iii) $|f(x)|^2 \leqslant f(1)f(xx^*)$ for all $x \in A$;

(iv) $f$ is continuous with $\|f\| = f(1)$.

Prove, conversely, that if $f$ is a continuous linear functional on $A$ with $\|f\| = f(1)$ then $f$ is positive. [N.B. You should assume, *without proof*, that, for every $x \in A$, $\mathrm{Sp}(x^*x) \subset \mathbb{R}^+$.]

Let $x \in A$ and let $\lambda \in \mathrm{Sp}\, x$; prove that there is a positive linear functional on $A$ such that both $\|f\| = 1$ and $f(x) = \lambda$.

Deduce that, for every non-zero $x \in A$ there is some positive linear functional with $f(x) \neq 0$.