UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 31 May, 2004 9 to 12

PAPER 1

NOETHERIAN ALGEBRAS

Attempt **THREE** *questions.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 State and prove the Hilbert Basis Theorem.

Prove that the complex group algebra $\mathbb{C}G$ of the infinite dihedral group $G = \langle X, Y : Y^{-1}XY = X^{-1}, Y^2 = 1 \rangle$ is left Noetherian.

Describe the centre R of $\mathbb{C}G$.

Show that $\mathbb{C}G$ is a finitely generated R-module and that the annihilator P in R of any simple left $\mathbb{C}G$-module M is a maximal ideal of R.

Deduce that M is a finite dimensional $\mathbb{C}$-vector space.

2 Let R be a commutative Noetherian ring and let M and N be finitely generated R-modules.

(i) Show that $M = 0$ if and only if the localisation $M_P = 0$ for all prime ideals P.

(ii) Let $\phi : M \to N$ be an R-module homomorphism. Show that ϕ is surjective if and only if $\phi_P : M_P \to N_P$ is surjective for all prime ideals P.

(iii) Show that the prime ideals Q_P of R_P (where P is a prime ideal of R) are in $1-1$ correspondence with the prime ideals Q of R contained in P.

(iv) Show that the set of associated primes in R_P of M_P

$$\mathrm{Ass}_{R_P}(M_P) = \{Q_P : Q \in \mathrm{Ass}_R(M), Q \leqslant P\}$$

for any prime ideal P.

3 Show that the first complex Weyl algebra $A_1(\mathbb{C})$ has no non-trivial zero divisors.

State Ore's Theorem and sketch its proof.

Show that $A_1(\mathbb{C})$ has a classical ring of left quotients which is a division ring.

4 Write an essay about the module theory of the nth complex Weyl algebra $A_n(\mathbb{C})$. You should include a proof of Bernstein's inequality.

5 Write an essay on Hochschild cohomology explaining in particular the significance of $H^1(R, R)$ and $H^2(R, R)$ for a ring R.