UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 30 May 2003 9 to 11

PAPER 78

SEISMIC WAVES

Any number of questions may be attempted.
Full marks can be obtained for two complete answers or their equivalence.

There are **four** *questions in total.*

The questions carry equal weight.

Candidates may use their lecture notes, any material handed out during the course and examples classes, and any hand-written or typed notes, taken from sources outside the lectures, which they have prepared themselves.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Describe and explain the transmission and reflection properties of one-dimensional seismic waves for a composite medium composed of many parallel uniform layers. You may draw on your own investigation of these properties and your experience using the spread-sheet program provided during the course for one-dimensional waves in such media imbedded between uniform half-spaces.

2 For two-dimensional SH waves, SH-wave energy is radiated equally in all azimuths from a line source consisting of a time-varying, two-dimensional delta-function force directed along the line of the source.

(i) Use the divergence theorem to establish the leading-order near-source spatial dependence of the SH-wave displacement directly from the SH-wave equation in a uniform isotropic medium.

(ii) Use ray theory to establish the general spatial dependence of the amplitude of leading-order wavefield discontinuities in a uniform isotropic medium. Can you reconcile this result and the result from part (i).

(iii) For the two-dimensional problem of two uniform isotropic media welded together at a plane interface with a line source in one of these media, calculate the position of each ray in the other medium as a function of the travel time and the component p of the slowness parallel to the interface. Only consider values of p that are smaller than $1/\beta_{\mathrm{max}}$ where β_{max} is the larger of the S wave speeds in the two media.

(iv) Use ray theory to deduce the amplitude of transmitted leading-order wavefield discontinuities in this case, again as a function of the travel time and p, and also only considering values of p that are smaller than $1/\beta_{\mathrm{max}}$.

3 For Love waves in a uniform isotropic superficial layer, with properties (ρ', β'), over a homogeneous isotropic half-space, with properties (ρ, β), an approximation that is sometimes used when $\beta' \ll \beta$ is to treat the half-space as though it is rigid. That is, the displacement is implicitly taken to be zero throughout the half-space by applying a boundary condition of zero displacement at the base of the superficial layer.

(i) For each of the possible guided modes in the case of SH waves in a uniform isotropic plane layer with a free-surface boundary condition at the top and rigid boundary condition at the bottom, deduce the dispersion relationships for phase velocity $c = \omega/k$ and group velocity $U = d\omega/dk$ as functions of frequency ω, analogous to the dispersion curves for Love waves shown by Hudson in his Figure 3.5 on page 75 where k is the wavenumber in the horizontal direction.

(ii) Estimate what the frequency ranges are where these dispersion relationships are acceptable approximations to the dispersion relationships for the full Love wave case when $\beta' \ll \beta$, relating these frequency ranges to the frequencies at which the group velocity has minima for the actual Love waves.

(iii) Now consider the case of a uniform anisotropic superficial layer with horizontal and vertical symmetry axes over a homogeneous isotropic half-space. In the superficial layer the stress-strain relationships are

$$\begin{pmatrix} \sigma_{yz} \\ \sigma_{xy} \end{pmatrix} = \begin{pmatrix} \rho'\beta_z'^2 & 0 \\ 0 & \rho'\beta_x'^2 \end{pmatrix} \begin{pmatrix} \partial u_y/\partial z \\ \partial u_y/\partial x \end{pmatrix} ,$$

where y is the direction of the SH wave displacement, x and z are the horizontal and vertical directions respectively, and β_x' and β_z' are the SH wave speeds in those directions. Explain how the first-principles derivation of the Love wave dispersion relationship, up to Hudson's equation (3.39), changes from the isotropic case. Also explain how the expressions obtained in parts (i) and (ii) of this question change, assuming that $\beta_x' \ll \beta$ and $\beta_z' \ll \beta$.

4 A simple model of the seismic source associated with the intrusion of magma into a dike below a volcano is that of a tensile crack across which there is a discontinuity $\left[u^{[n]}\left(\mathbf{x},\mathbf{t}\right)\right]\mathbf{n}\left(\mathbf{x}\right)$ in the displacement vector $\mathbf{u}\left(\mathbf{x},\mathbf{t}\right)$, where $u^{[n]}\left(\mathbf{x},\mathbf{t}\right) = \mathbf{u}\left(\mathbf{x},\mathbf{t}\right)\cdot\mathbf{n}\left(\mathbf{x}\right)$ is the component of $\mathbf{u}\left(\mathbf{x},\mathbf{t}\right)$ in the direction of the normal $\mathbf{n}\left(\mathbf{x}\right)$ to the crack: in essence, the dike widens as the magma is intruded.

(i) What is the body force that is equivalent to such a seismic source in an isotropic elastic medium?

(ii) Show that at each point $\mathbf{x}$ of the crack this body force is composed of 3 dipoles, two of which are equal in magnitude and are in the plane of the crack, with the third dipole in the direction of $\mathbf{n}$ being larger than the other two dipoles.

(iii) Using the expression

$$G_i^k\left(\mathbf{x},\mathbf{y},\mathbf{t}\right) = \frac{\widehat{x}_i\widehat{x}_k}{4\pi\rho\alpha^2 R}\delta\left(t-\frac{R}{\alpha}\right) + \frac{1}{4\pi\rho\beta^2 R}\left(\delta_{ik}-\widehat{x}_i\widehat{x}_k\right)\delta\left(t-\frac{R}{\beta}\right) + \frac{1}{4\pi\rho R^3}\left(3\widehat{x}_i\widehat{x}_k-\delta_{ik}\right)t\left[H\left(t-\frac{R}{\alpha}\right)-H\left(t-\frac{R}{\beta}\right)\right]$$

for the elastic Green's function in an unbounded isotropic medium, where

$$R = |\mathbf{x}-\mathbf{y}| \quad \text{and} \quad \widehat{\mathbf{x}} = \frac{(\mathbf{x}-\mathbf{y})}{R},$$

derive the expression for the displacement field generated by a small area of the tensile crack source in such a medium.

(iv) In this expression identify the purely transient P and S wave displacements that decay as $1/R$ and the final static displacement field after magma intrusion has stopped, and sketch how at constant R these displacements depend on the direction from the source for the case of a Poisson solid for which $\lambda = \mu$ and $\alpha^2 = 3\beta^2$.