UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 29 May 2003 1.30 to 3.30

PAPER 77

SOLIDIFICATION OF FLUIDS

ALL *questions may be attempted.*
Full marks can be obtained by substantially complete answers to **TWO** *questions.*

There are **three** *questions in total.*

The questions carry equal weight.

Candidates may bring into the examination any lecture notes made during the course and any materials that were distributed during the course.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 This question concerns a polynya, which is an open stretch of ocean in polar regions in which small ice crystals form owing to heat losses to the cold atmosphere and are carried in suspension in a turbulent boundary layer away from a coastline, driven by strong off-shore winds.

Imagine that the ocean has salinity C_∞ and temperature T_∞ equal to the liquidus temperature $T_L(C_\infty) = -mC_\infty$, where m is the constant liquidus slope. Suppose that the wind exerts a constant stress S and that there is a constant heat flux F from ocean to atmosphere. Steady equations governing the depth $b(x)$ of the turbulent boundary layer, the mean horizontal velocity in it $u(x)$ and its mean temperature $T(x)$, bulk composition $\bar{C}(x)$ and volume fraction of ice crystals $\phi(x)$ are

$$\frac{d}{dx}(\bar{\rho} b u^2) = S + (\bar{\rho} - \rho_\infty) g b \frac{db}{dx}, \tag{1}$$

$$\frac{d}{dx}\left[\{c_p(T - T_\infty) - L\phi\}\, bu\right] = -F, \tag{2}$$

$$\frac{d}{dx}\left[(\bar{C} - C_\infty) bu\right] = 0, \tag{3}$$

where $\bar{\rho}(x) = \phi\rho_s + (1-\phi)\rho_\infty$ is the bulk density of the suspension, ρ_s is the density of ice, ρ_∞ is the density of the ocean water, g is the acceleration due to gravity, c_p is the specific heat capacity, L is the specific latent heat and x is the horizontal coordinate measured from the coastline.

What physical laws do these equations represent?

Describe the physical meaning of each term in equation (1) and derive equations (2) and (3) by applying conservation laws over suitable control volumes. What boundary conditions should be applied to them?

It may be assumed that one quarter of the work done by the wind stress is converted into potential energy, so that

$$\tfrac{1}{4} S u = -(\bar{\rho} - \rho_\infty) g b u \frac{db}{dx}, \tag{4}$$

and that the turbulent boundary layer is in local thermodynamic equilibrium, so that

$$\bar{C} = (1 - \phi) C, \quad \text{where} \quad C = -T/m \tag{5}$$

is the salinity of the interstitial brine.

Eliminate T and $\bar{C}$ between equations (2), (3) and (5) and find a similarity solution of the remaining equations for b, u and ϕ valid when $\phi \ll 1$. [*You should ignore all density variations except insofar as they modify the bouyancy.*]

How would equations (1)–(5) be modified if account were taken of the upwards settling of ice crystals relative to the dense interstitial brine? Comment with reasons on whether the predicted rate of ice production would be larger or smaller in this case.

2 A certain binary alloy has a simple phase diagram with linear liquidus $T_L = -mC$ and constant segregation coefficient k_D, where C is the solute concentration.

At time $t = 0$, pure liquid of uniform concentration $C = 0$ and temperature T_l, occupying the region $x > 0$, is brought into contact with solid of uniform concentration C_s and temperature T_s, occupying $x < 0$. Assume that subsequently there is a single phase boundary at $x = a(t)$ separating liquid from solid and write down a similarity solution for the temperature and solute concentration fields. Show that

$$\frac{L}{c_p} = \frac{T_a - T_s}{-F(-\delta\lambda)} - \frac{T_l - T_a}{F(\delta\lambda)} \tag{1}$$

and

$$(1 - k_D)C_a = \frac{k_D C_a - C_s}{-F(-\lambda/\epsilon)} + \frac{C_a}{F(\lambda)}, \tag{2}$$

where L and c_p are the specific latent heat and heat capacity of the alloy, T_a and C_a are the temperature and liquid concentration at the phase boundary, $\delta^2 = D_l/\kappa$ and $\epsilon^2 = D_s/D_l$ are ratios between solute diffusivities D_l and D_s in the liquid and solid phases respectively and the diffusivity of heat κ, assumed independent of phase, and the function

$$F(z) = \sqrt{\pi}\, z\, e^{z^2} \operatorname{erfc}(z),$$

while $\lambda = a(t)/2\sqrt{D_l t}$. What relationship is there between T_a and C_a?

Consider the cases in which $\lambda = \mathrm{O}(1)$ while $\delta \ll 1$ and $\epsilon \ll 1$. Linearize equation (1) to find an expression for T_a correct to $\mathrm{O}(\delta)$ and, from equation (2), sketch a plot of C_a as a function of λ. [*You may find it helpful to consider separately the cases* $\lambda > 0$ *and* $\lambda < 0$.]

Hence, or otherwise, show that if $L/c_p > 2mC_s/\delta$ then there is a unique solution for λ, while for smaller values of L/c_p there is a range of values of $T_l + T_s$ for which there are multiple solutions. You should determine the multiplicity but need not determine the range of values of $T_l + T_s$. Sketch a plot of λ as a function of $T_l + T_s$ in the latter case.

For the case $T_l + T_s > 0$, sketch plots of T and C as functions of x, indicating the relative sizes of the various boundary layers, and sketch the trajectory of T and C in the phase diagram, parameterized by x. Use your sketch to show that it is possible for the solid to become superheated (have a temperature greater than its solidus) and discuss briefly what you would expect to happen in such a case.

[*Note that*

$$F(z) \to 1 \quad as \quad z \to \infty$$
$$F(z) \to -\infty \quad as \quad z \to -\infty\,.]$$

3 Starting from the van-der-Waals force between two molecules, show that the force of attraction between a sphere of one material and a semi-infinite region of another is approximately

$$\frac{A_{12}\,a}{6d_0^2}, \tag{1}$$

where a is the radius of the sphere, $d_0 \ll a$ is the closest distance between the sphere and the planar surface of the other material, and A_{12} is the Hamaker constant for the two materials.

If there is liquid surrounding the sphere then there can be a *repulsive* force between the sphere and the plane of magnitude given by (1) with A_{12} replaced by an effective Hamaker constant A for the three-component system.

Consider the unidirectional solidification of water at constant rate V driven by a constant temperature gradient G. Show that a spherical particle suspended in the water can be pushed steadily at a distance d_0 in front of the ice–water interface (assumed planar), where

$$V = \frac{A}{36\pi\mu a d_0},$$

and μ is the dynamic viscosity of the water. [*Hint: Use lubrication theory and make a parabolic approximation for the surface of the sphere.*]

Interfacial premelting causes the ice–water interface to be deformed by the presence of the particle. Show that when the base of the sphere is a distance $h = a(1 - \cos\theta_0)$ below the surface of the ice (as shown below), the sphere can still be pushed steadily at a rate

$$V = \frac{A}{9\pi\mu a^2}\left[\frac{2 - 3\cos\theta_0 + \cos^3\theta_0}{3 - 8\cos\theta_0 + 6\cos^2\theta_0 - \cos^4\theta_0}\right].$$

You should assume that the premelted liquid film has a thickness $d(\theta) \ll h$ (not drawn to scale in the figure), that the temperature field is unaffected by the presence of the phase boundary or the particle and that the depression of the melting temperature due to curvature is negligible.

[*You may quote the result that the volume flux per unit area in a thin film of thickness d is equal to $-\dfrac{d^3}{12\mu}\dfrac{\partial p}{\partial x}$, where x is the streamwise coordinate and p is the fluid pressure.*]