UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2003 1.30 to 3.30

PAPER 76

ENVIRONMENTAL FLUID DYNAMICS

Attempt **BOTH** *questions from Section A,*
ONE *question from Section B*
and **ONE** *question from Section C.*

There are **two** *questions in Section B and* **two** *questions in Section C.*

Questions in Section A carry half the weight of questions in Section B and C.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Section A

Answer both questions from Section A.

1 A brewery produces a special beer in which bubbles are only formed while the beer is being drawn. The bubbles, described by the void fraction $\phi(z,t)$, are all of the same size and rise with a velocity given by $v = (1-\phi)V_s$, where V_s is the rise velocity of an isolated bubble.

(a) A standard cylindrical tankard of height H is filled with a uniform distribution of bubbles $\phi = \phi_0$. Describe the evolution of the bubble field, assuming the fluid velocities are small compared with the rise velocity of the bubbles. Give an expression for the mean void fraction $\Phi(t)$ of bubbles within the tankard.

(b) The brewery designs a new tankard of height H with a radius r described by $\frac{dr}{dz} = -\beta r$. Assume there is no turbulence and the bubble distribution remains horizontally uniform. Derive the continuity equation for the void fraction in this tankard. Determine the equation of the characteristics and an expression for the behaviour of ϕ along these characteristics.

(c) Determine $\eta(t)$, the distance a bubble has risen by time t.

2 Consider a fluid layer of depth h between two horizontal, rigid boundaries. The fluid has thermal conductivity k, thermal diffusivity κ, kinematic viscosity ν and thermal expansion coefficient β, all assumed constant. The acceleration due to gravity is g.

(a) Define the Rayleigh number and describe the 4/3's heat transfer 'law' for the case when the lower boundary is maintained at a temperature of $T_o + \Delta T$ and the upper boundary is maintained at a temperature of T_o with $\Delta T > 0$.

(b) Suppose that the initial temperature of the fluid layer is $T = T_o + \Delta T$ ($\Delta T > 0$) and from time $t = 0$, the upper boundary is maintained at a temperature T_o while the lower boundary is perfectly insulating to heat. Determine the temperature of the fluid layer, $T(t)$, as a function of time, stating all assumptions. How would the cooling process change if the upper boundary was insulating and the lower boundary was maintained at a temperature of T_o.

Section B

Answer only one question from Section B.

3 Consider the steady hydraulic flow of a fluid of density ρ_1 in a channel of width $b(x) = 1 + (x/L)^2$ with a flat bottom and rectangular cross-section. Above this flow is a stationary fluid with density ρ_0. Under the Boussinesq approximation, the Bernoulli potential in the lower layer is given by $B = \frac{1}{2}u^2 + g'h$ where $u(x)$ is the velocity, $h(x)$ is the depth and g' the reduced gravity.

(a) Discuss the 'shallow water' assumption. Derive the specific energy function E and show how $\partial E/\partial h$ is related to the Froude number F. Explain what is meant by the terms 'subcritical', 'supercritical' and 'hydraulic control'. Sketch the possible configurations the flow may have and describe these in terms of the propagation of long waves.

(b) For a flow that is hydraulically controlled with depth H far upstream, show that the hydraulic control is located at $x = 0$ and determine the depth and velocity at this point.

(c) A hydraulic jump forms where $b = (32/27)^{1/2}$. Determine the conditions at the jump, stating any assumptions made.

(d) In general, there will be a loss of energy from the flow as it moves downstream. Describe briefly the processes that cause this. Suppose the energy loss causes the Bernoulli potential to diminish along the channel as $\partial B/\partial x = -\alpha$ in the neighbourhood of $x = 0$. This causes the hydraulic control to move. Determine the new location for the control.

4 A channel with a uniform parabolic cross-section (width $b(z) = \alpha z^2$) is dug across a plane to provide irrigation for a new farm. The channel is lined with polythene to prevent seepage.

(a) Derive the continuity equation for a shallow water flow in this channel. State the momentum equation in an appropriate form for the flow of water along the channel. You may neglect drag terms. Show that the system is hyperbolic. Determine the wave speed c, the slope λ of the characteristics, and the quantity conserved along the characteristics.

(b) A dam is used to block the channel half way along its length, and the channel is filled on the upstream side to a depth H. However, the dam collapses at $t = 0$ and a front propagates downstream. In the absence of drag, how fast does the front propagate? Determine and sketch the depth and velocity fields. Describe briefly how bottom drag will affect the form of the front.

(c) To prevent the reservoir feeding the irrigation channel from draining, a truck load of sand is dumped in the channel a distance $\ell \gg H$ upstream of the dam just before it collapses. Describe, with reference to sketches and an $x - t$ diagram, the evolution of the flow.

(d) A week later it is discovered that water is leaking through the temporary sand dam so that the depth of water downstream is η. The temporary dam has a length L, permeability k and porosity ϕ. Determine the depth profile and flux for the flow in the porous media, stating any assumptions made. You may neglect surface tension effects.

(e) Closer inspection showed that two different loads of sand, with different permeabilities, were used to form the dam. These were arranged with a lower layer of depth ζ and permeability k_1. The upper layer with permeability k_2 extends above the water surface. For the case where $\zeta < \eta$, determine the flux through the dam.

Section C

Answer only one question from Section C.

5 The governing equations for a steady Boussinesq axisymmetric plume rising in an unstratified environment are

$$\frac{d}{dz}(wb^2) = 2\alpha wb,$$

$$\frac{d}{dz}(w^2b^2) = g'b^2,$$

$$\frac{d}{dz}(g'wb^2) = 0,$$

where z is the height above the source, b is the plume radius and top-hat profiles are assumed (flow quantities independent of radius for $r \leq b$ and zero for $r > b$). Here α is the entrainment coefficient, w is the vertical velocity and g' is the reduced gravity.

(a) Solve this system of equations for b, w and g' for the case of a pure plume with buoyancy flux B_o and zero initial flux of volume and momentum. Discuss the Boussinesq approximation for steady state plumes. When does the approximation break down?

Now consider the case of a steady non-Boussinesq axisymmetric plume due to a point source of heat in a uniform ambient of temperature T_o. The temperature of the plume is T and its density is $\rho = \frac{\beta}{T}$ where β is a constant. (The density in the ambient is $\rho_o = \frac{\beta}{T_o}$). Non-Boussinesq plumes can be modelled using

$$\frac{dQ}{dz} = 2\alpha wb(\rho\rho_o)^{1/2},$$

$$\frac{dM}{dz} = g(\rho_o - \rho)b^2,$$

$$\frac{dF}{dz} = 2c_p\alpha wb(\rho\rho_o)^{1/2}T_o,$$

where $Q = \rho wb^2$ is the mass flux in the plume, $M = \rho w^2b^2$ is the momentum flux in the plume and $F = c_p\rho wb^2T$ is the heat flux in the plume, g is the acceleration due to gravity and c_p is the specific heat (assumed constant). The local buoyancy flux is defined as $B = g\frac{\rho_o - \rho}{\rho_o}wb^2$.

(b) Show that B is conserved in a uniform ambient.

(c) By writing w, b and ρ in terms of Q, M and B show that

$$\frac{dQ}{dz} = 2\alpha M^{1/2}\rho_o^{1/2}, \quad \frac{dM}{dz} = \frac{BQ\rho_o}{M}.$$

Hence determine Q and M and show that ρ can be written in the form

$$\rho = \frac{\rho_o}{1 + (\frac{z_b}{z})^{5/3}}$$

where z_b is a constant which is to be determined.

(d) Determine b and $g' = g\frac{\rho_o - \rho}{\rho_o}$ and comment on the physical significance of z_b.

6 Consider steady natural ventilation in an enclosure of depth H with a lower level opening of area A_b and an upper level opening of area A_t. A two-layer stratification develops (see Figure 1(a)) due to a point source of buoyancy of strength B_1 at the base of the enclosure.

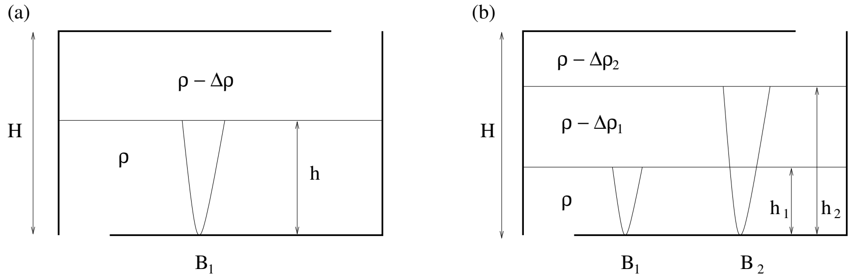

Figure 1: Steady state ventilation due to (a) one and (b) two localized buoyancy sources.

(a) Determine the volume flow rate through the upper and lower level openings and hence determine an implicit relationship for the height of the density interface h_1, stating all assumptions. You may assume that the volume flux of a plume in an infinite, uniform ambient is given by $Q_p = CB^{1/3}z^{5/3}$ where C is a constant and z is the distance from the source.

Now consider the three layer stratification in Figure 1(b) which develops due to the presence of non-interacting plumes rising from sources at the base of the enclosure of strength B_1 and B_2 with $B_1 < B_2$. The density in the middle layer is the same as the density in plume 1 at height h_1 and the density in the upper layer is the same as the density in plume 2 at height h_2.

(b) What is the volume flow rate through the upper and lower level openings?

(c) Show that

$$\frac{A^*}{H^2C^{3/2}} = \frac{(1+\psi^{1/3})^{3/2}}{(1+\psi)^{1/2}}\left(\frac{(h_1/H)^5}{1-\frac{h_1}{H}-\frac{1-\psi^{2/3}}{1+\psi}\frac{h_2-h_1}{H}}\right)^{1/2}$$

and

$$\frac{\Delta\rho_1}{\Delta\rho_2} = \frac{\psi+\psi^{2/3}}{1+\psi}$$

where A^* is the suitably defined reduced area of the openings and $\psi = B_1/B_2$. Discuss the limits $\psi = 0$ and $\psi = 1$.

(d) When plume 2 crosses the interface at h_1 it can no longer be modelled as a plume rising in a uniform environment. Discuss why this is, and describe (without solving any equations) how the plume could be modelled after it crosses the interface, stating all assumptions.

(e) If it is assumed that the volume flux in plume 2 at height h_2 is $Q_2(h_2) = C(B_1+B_2)^{1/3}h_2^{5/3}$, show that

$$\frac{h_2}{h_1} = \left(\frac{1+\psi^{1/3}}{(1+\psi)^{1/3}}\right)^{3/5}.$$