UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 4 June 2003 9 to 11

PAPER 75

PHYSIOLOGICAL FLUID DYNAMICS

Attempt **TWO** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

Candidates may use their lecture notes, any material handed out during the course and examples classes, and any hand-written or typed notes, taken from sources outside the lectures, which they have prepared themselves.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Measurements of pressure and flow-rate wave forms at fixed sites in arteries show that, within a single cardiac cycle, the time t_1 at which the pressure is maximum is later than the time t_2 at which the flow rate is maximum. Conventional measurements at peripheral sites show that the time difference $t_1 - t_2$ decreases with increasing age. However, recent measurements in the ascending aorta indicate that $t_1 - t_2$ increases with age.

You are invited to seek to explain all the above findings by modelling the propagation of the pulse wave and its reflection at the aortic bifurcation, which is known to be a site at which the net cross-sectional area of the vessels decreases. It is also known that arteries become stiffer with age, but the geometry of the aortic bifurcation is relatively unaffected by age. Be explicit about all assumptions and approximations in your model.

[Hint: it is suggested that the peripherally-travelling part of the pulse wave in the aorta is modelled as a cosine wave, in which the pressure is

$$p = P_I \cos\left[\omega\left(t - x/c_1\right)\right],$$

where x, t are longitudinal coordinate and time, P_I is a constant amplitude, ω is angular frequency and c_1 is wave speed.]

2 A collapsible tube of finite length L rests on a rigid planar surface inclined at an angle θ to the horizontal. The tube elasticity is described by a tube law in which the internal pressure is given by

$$p = P_0 + \frac{1}{2}\rho c_0^2 A^2/A_0^2,$$

where P_0 is a constant pressure, ρ is fluid density, A is the tube's cross-sectional area, A_0 is a constant area and c_0 is a constant speed. When fluid flows steadily downhill the viscous resistance can be represented by a term $-\rho R_0 R(A/A_0)Au$ in the one-dimensional momentum equation, where R_0 is constant and $R(\alpha)$ is a dimensionless function such that $dR/d\alpha < 0$, and u is the fluid velocity along the tube.

Fluid enters the tube at $x = 0$ with flow rate $c_0 A_0 q$ and the cross-sectional area is $A_0\alpha_1$.

(a) Show that this represents a stable, steady-state solution of the governing equations as long as both

$$qR(\alpha_1) = \beta \quad (\text{defined by} \beta = \frac{g\sin\theta}{c_0 A_0 R_0})$$

and

$$\alpha_1 < q^{1/2}.$$

Explain the significance of this inequality.

(b) At the downstream end of the tube, $x = L$, the cross-sectional area is $A_0\alpha_3$ where $\alpha_3 > q^{1/2}$. Explain how this can normally be achieved through the occurrence of an elastic jump at some position x_s, such that $0 \leqslant x_s \leqslant L$. For the case in which $R(\alpha) \equiv \alpha^{-\gamma}(\gamma > 0)$ and $\beta > q^{1-\gamma/2}$, show, by analysing both the jump conditions (neglecting gravity and viscous resistance in the jump itself) and the flow downstream of the jump, that the value of x_s can in principle be determined by solving the following three equations:

$$\alpha_1^\gamma = q/\beta$$

$$\alpha_2^3 + \alpha_1\alpha_2^2 + \alpha_1^2\alpha_2 - \frac{3q^2}{\alpha_1} = 0$$

$$\int_{\alpha_2}^{\alpha_3} \frac{\alpha^4 - q^2}{\alpha^3 - (q/\beta)\alpha^{3-\gamma}} d\alpha = \frac{g\sin\theta}{c_0^2}(L - x_s).$$

(c) Hence show that, in the limit $\beta \to \infty$ with $q = O(1)$, and for the case $\gamma = 2$

$$\frac{2g\sin\theta}{c_0^2}(L - x_s) \approx \alpha_3^2 - 3^{2/3}q\beta^{1/3},$$

provided that the right hand side is positive and less than $gL\sin\theta/c_0^2$.

3 Fully-developed steady flow along an annular channel of width $\hat{h}$ has a velocity profile

$$\hat{u} = \hat{U} u_0(y), \quad 0 \leqslant y \leqslant 1,$$

where the radial coordinate is $\hat{r} = \hat{h}(R+y)$, and in which the magnitudes of the shear-rate on the two walls are different, i.e.

$$u_0'(0) = \gamma_0, \quad u_0'(1) = -\gamma_1, \quad \gamma_0 > \gamma_1 > 0$$

and $u_0(0) = u_0(1) = 0$. Here $\hat{U}$ is a velocity scale, and R, γ_0, γ_1 are dimensionless constants. Do NOT calculate $u_0(y)$ explicitly.

Axisymmetric perturbations to this flow can be analysed in the same way as for a planar channel, apart from the fact that the continuity equation is $\hat{u}_{\hat{x}} + \frac{1}{\hat{r}}(\hat{r}\hat{v})_{\hat{r}} = 0$ where $(\hat{x}, \hat{r})$ are cylindrical polar coordinates with corresponding velocity components $(\hat{u}, \hat{v})$, and the viscous terms are also modified.

The inner wall of the channel, $y = 0$, is subjected to a time-dependent indentation,

$$y = \epsilon F(x, t),$$

where $F(x,t) = 0$ for $x \leqslant 0$ and $x \geqslant 1$, $x = \hat{x}/\lambda h$, $t = \omega \hat{t}$ ($\hat{t}$ is dimensional time), ω is characteristic frequency and λ, ϵ are dimensionless quantities such that

$$\lambda \gg 1, \quad \epsilon \ll 1.$$

The Reynolds number is $Re = \hat{U}\hat{h}/\nu \gg 1$; the Strouhal number is $St = \omega \hat{h}/\hat{U} \ll 1$.

(i) Explain carefully the relative orders of magnitude of the parameters $\epsilon, \lambda, Re, St$ that

(a) permit the flow to be analysed as an inviscid core with two boundary layers on the walls, of dimensionless thickness $\delta \ll \epsilon$; and

(b) allow the dimensionless longitudinal velocity in the core to be written as

$$u = u_0(y) + \epsilon \frac{A(x,t)}{R+y} u_0'(y) + \epsilon^2 u_2(x,y,t) + \ldots.$$

Show that $A(x,t)$ satisfies the following partial differential equation, as long as the boundary layer thickness remains of $O(\delta)$ everywhere:

$$\sigma A_{xxx} - \beta \alpha_1 A_t - \alpha_2 A A_x = \beta \gamma_0 F_t + \gamma_0^2 F F_x + \frac{\gamma_0^2}{R}(AF)_x, \qquad (*)$$

where

$$\beta = \lambda St \epsilon^{-1}, \quad \sigma = \lambda^{-2} \epsilon^{-1} \int_0^1 \frac{u_0^2(y)}{R+y} dy, \quad \alpha_1 = \frac{\gamma_0}{R} + \frac{\gamma_1}{R+1}, \quad \alpha_2 = \frac{\gamma_0^2}{R^2} - \frac{\gamma_1^2}{(R+1)^2}.$$

(ii) Deduce that small amplitude (linear) sinusoidal waves can propagate downstream (and not upstream), with group velocity equal to three times the phase velocity.

For regions in which $F = 0$, investigate nonlinear waves of permanent form, given by $A(\xi)$ where $\xi = x + ct$, and such that A and all its derivatives tend to zero smoothly as $\xi \to \pm\infty$. By integrating equation (*) twice in that case, show (for example graphically) that such waves can propagate upstream (but not downstream), with $A < 0$ and $|A|_{max} = 3\beta\alpha_1 c/\alpha_2$.

[The above is a model of a cardiac assist device consisting of a balloon mounted axisymmetrically on a catheter in the aorta, and inflated periodically.]

4 A model for a red blood cell passing steadily down an otherwise plasma-filled capillary consists of an axisymmetric elastic body of unstressed radius $r_0(x), -L \leqslant x \leqslant L$, where x is the longitudinal coordinate, surrounded by incompressible viscous fluid contained in a rigid cylinder of radius a. Near $x = 0$, $r_0(x)$ is approximately parabolic:

$$r_0(x) \approx r_{00} - \frac{1}{2}\kappa x^2,$$

where $\kappa > 0$ and r_{00} may be assumed to be greater than a. The cell elasticity is modelled linearly, so that the pressure in the lubricating film of fluid around the cell is given by

$$p = p_0 + \alpha[r_0(x) - r(x)],$$

where p_0, α are positive constants and $r(x)$ is the actual cell radius. The "cell" moves in the $+x$ direction with speed U, and the pressure in the plasma behind the cell exceeds that in front by Δp. The goal is to find a relationship between Δp and U, on the assumption that inertia is negligible.

Taking axes fixed in the cell, use lubrication theory to analyse the flow in the lubricating film, showing in particular that

$$\frac{dp}{dx} = -\frac{6\mu U}{h^2} + \frac{12\mu Q}{h^3},$$

where $h(x)$ is the film thickness, μ is the fluid viscosity and $-2\pi a Q$ is the (unknown) volume flow rate of fluid past the cell in the $+x$ direction. Write down boundary conditions at $x = \pm L$. What further conditions must be imposed to complete the formulation of the problem?

Setting $h = (2Q/U)H$ and $x = \left(\frac{2Q/U}{\kappa}\right)^{1/2} X$, show that the problem can be reduced to:

$$\frac{dH}{dx} + \lambda\left(\frac{1}{H^2} - \frac{1}{H^3}\right) = X \tag{1}$$

with

$$H(-\tilde{L}) - H(\tilde{L}) = C\lambda \int_{-\tilde{L}}^{\tilde{L}} \left(\frac{4}{3H} - \frac{1}{H^2}\right) dX, \tag{2}$$

where

$$C = \frac{2Q}{Ua}, \quad \lambda = \frac{6\mu U}{\alpha\kappa^{1/2}(2Q/U)^{5/2}}, \quad \tilde{L} = L\left(\frac{\kappa}{2Q/U}\right)^{1/2}.$$

Then

a) express Δp as a multiple of the left hand side of equation (2);

b) explain why self-consistency of the model requires $C \ll 1$;

c) seek a solution in the limit of small λ and large $\tilde{L}$, of the form

$$H = H_0(x) + \lambda H_1(x) + \ldots.$$

Show that

$$H_0 = \frac{1}{2}\left(b^2 + X^2\right),$$

where

$$\frac{1}{b^2} \approx \frac{2}{3}(1+C)\,.$$

Deduce that

$$\Delta p \approx \frac{8\pi(2/3)^{1/2}\mu U}{a\kappa^{1/2}(2Q/U)^{1/2}}$$

so that the dependence of Δp on U can be obtained, finally, from a statement of the relationship between p_0 and the downstream pressure $p(L)$.

[You will need the integrals

$$\int_{-\infty}^{\infty} \frac{dX}{(b^2+X^2)^n} = \frac{a_n \pi}{b^{2n-1}}, n=1,2,3, \quad \text{where} \quad a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{3}{8}.$$]