UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 9 June 2003 1.30 to 4.30

PAPER 74

WAVE THEORY

Attempt no more than **THREE** *questions. Little credit will be given for fragments.*

There are **five** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write an essay on **one** of the following topics.

Either: The use of transform methods to investigate the behaviour of a fluid-loaded elastic plate undergoing time-periodic line forcing. You may assume that the motion of the plate, which lies along the x-axis when at rest, is governed by the equation

$$m\frac{\partial^2\eta}{\partial t^2} + \beta\frac{\partial^4\eta}{\partial x^4} = F(x,t) - p(x,0,t)$$

where $\eta(x,t)$ is the transverse displacement, m the mass per unit length, β the bending stiffness, F the forcing and $p(x,y,t)$ the fluid pressure.

Include in your essay a discussion of at least the following points: calculation of the plate displacement; the far-field in the fluid; the possible presence of other waves (no detailed calculations are required); the difference between subsonic and supersonic wave components; and the power radiated away from the plate in the y-direction.

Or: The use of the Wiener-Hopf technique to solve the Sommerfeld problem of diffraction of a plane wave by an edge, i.e. solve

$$(\nabla^2 + k_0^2)\phi = 0$$

subject to

$$\frac{\partial\phi}{\partial y} + \frac{\partial\phi_i}{\partial y} = 0 \quad \text{on} \quad y = 0, x < 0\ ,$$

where

$$\phi_i = \exp(-\mathrm{i}k_0 x\cos\theta_0 - \mathrm{i}k_0 y\sin\theta_0 - \mathrm{i}\omega t)$$

is the incident potential, $\phi(x,y)\exp(-\mathrm{i}\omega t)$ is the scattered potential and $k_0 = \omega/c_0$.

Your essay should include a derivation of expressions for the geometrical optics field and the far-field form of the diffracted field.

[*You may quote without proof the result*

$$\int f(k)\exp(\mathrm{i}kr\cos\theta - \gamma r\,|\sin\theta|)\,\mathrm{d}k \sim \sqrt{2\pi k_0/r}\, f(k_0\cos\theta)\,|\sin\theta|\exp(\mathrm{i}k_0 r - \mathrm{i}\pi/4)$$

as $r\to\infty$, where $\gamma^2 = k^2 - k_0^2$ and the integration is taken along the steepest descent contour (which crosses the real k-axis at $k_0\cos\theta$ and $k_0\sec\theta$). You should give a clear definition of the branch cuts which define γ.]

2 An incompressible inviscid fluid flows along a pipe with circular cross-section of radius a. The basic flow is $U(r)\hat{\mathbf{z}}$, where r is the cylindrical radial coordinate and $\hat{\mathbf{z}}$ is a unit vector along the axis of the pipe. Considering *only* axisymmetric disturbances, obtain an equivalent of Rayleigh's stability equation in the form

$$\big(U-c\big)\big(r(\phi'/r)'-k^2\phi\big)-r(U'/r)'\phi=0,$$

where $\phi(r)$ is the mode shape of the disturbance, k the axial wavenumber, c the wave speed, and a prime denotes differentiation with respect to r. What are the relevant boundary conditions?

For real values of k, is there an equivalent of Rayleigh's inflection-point theorem?

Now allow complex k, and consider Poiseuille flow in which $U(r)=U_0(1-r^2/a^2)$, where U_0 is a constant. Assume that $U \neq c$ for all r. By considering $r^{-n}\phi$ for a suitable choice of n, or otherwise, find an explicit expression for the disturbance mode shape.

How satisfactory are these results, and why? In particular, how useful are they for establishing stability criteria? Might including viscous terms make any difference? (No calculations are required, but you should indicate very briefly what steps might be involved and how your results might be affected.)

[*Hints: you may assume the following results in this question.*

Let $\mathbf{u}$ and $\mathbf{v}$ be arbitrary vectors where $\mathbf{u}=u_r\hat{\mathbf{r}}+u_\theta\hat{\boldsymbol{\theta}}+u_z\hat{\mathbf{z}}$ and $\mathbf{v}=v_r\hat{\mathbf{r}}+v_\theta\hat{\boldsymbol{\theta}}+v_z\hat{\mathbf{z}}$, in which $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{z}}$ are appropriate unit basis vectors in cylindrical polar coordinates (r,θ,z). Then

$$\mathbf{v}\,.\,\nabla\mathbf{u}=\Big\{\mathbf{v}\,.\,\nabla u_r-\frac{v_\theta u_\theta}{r}\Big\}\hat{\mathbf{r}}+\Big\{\mathbf{v}\,.\,\nabla u_\theta+\frac{v_\theta u_r}{r}\Big\}\hat{\boldsymbol{\theta}}+\Big\{\mathbf{v}\,.\,\nabla u_z\Big\}\hat{\mathbf{z}}$$

where, for any scalar $\Phi(r,\theta,z)$,

$$\nabla\Phi=\frac{\partial\Phi}{\partial r}\hat{\mathbf{r}}+\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\boldsymbol{\theta}}+\frac{\partial\Phi}{\partial z}\hat{\mathbf{z}}.$$

The Stokes streamfunction $\Psi(r,z,t)$ for axisymmetric flow corresponds to a velocity field

$$\mathbf{u}=-\frac{1}{r}\frac{\partial\Psi}{\partial z}\hat{\mathbf{r}}+\frac{1}{r}\frac{\partial\Psi}{\partial r}\hat{\mathbf{z}}.$$

The modified Bessel equation is

$$z^2y''+zy'-(z^2+\nu^2)y=0,$$

where ν is a constant. Its general solution is $y=AI_\nu(z)+BK_\nu(z)$, where A and B are arbitrary constants and I_ν, K_ν are the modified Bessel functions of order ν. For the value or values of ν relevant to this problem, as $z\to 0$, $I_\nu(z)\sim(\frac{1}{2}z)^\nu/\nu!$ while $K_\nu(z)\sim\frac{1}{2}(\nu-1)!(\frac{1}{2}z)^{-\nu}$; and the roots of I_ν all lie on the imaginary axis.]

3 (a) In a triangular jet of half-width a, varicose instability waves of wavenumber k and frequency ω have dispersion relation

$$\mathscr{D}(k,\omega) \equiv 2\omega + \mathrm{e}^{-2ka} - 1 = 0.$$

Can these modes exhibit absolute instability, convective instability, or neither? Illustrate your answer by explicit reference to the Briggs–Bers method for ensuring causality, in the complex k- and ω-planes.

(b) When $ka \ll 1$, approximate the dispersion relation by

$$\mathscr{D}(k,\omega) \approx \mathscr{D}^{\text{approx}}(k,\omega) \equiv 2\omega - 2ka + 2k^2a^2.$$

Verify that the model equation

$$\frac{\partial A}{\partial t} = -a\frac{\partial A}{\partial x} - ia^2\frac{\partial^2 A}{\partial x^2}$$

has an identical dispersion relation. Use this equation from now on to model the jet.

Consider a jet of slowly varying width, so that a is now a function of the slow spatial scale $X = \varepsilon x$ where $\varepsilon \ll 1$. The jet is subjected to time-periodic forcing of frequency Ω at the origin, where Ω is small, and responds with a wave of wavenumber

$$k^+(\Omega; X) = \frac{1 - \sqrt{1-4\Omega}}{2a(X)} \approx \frac{\Omega}{a(X)}$$

downstream of the forcing. By using a multiple-scales WKBJ-type approximation, or otherwise, show that the amplitude envelope of the wave develops like $a^{-\Omega}$.

4 Starting from the equations of mass and momentum conservation, derive Lighthill's equation for the acoustic density fluctuation $\rho'(\mathbf{x}, t)$ in the form

$$\frac{\partial^2 \rho'}{\partial t^2} - c_0^2 \nabla^2 \rho' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j} \ , \tag{1}$$

where T_{ij} is to be determined.

(i) Using equation (1), together with the free-space Green's function for the wave equation,

$$G(\mathbf{x}, t) = \frac{\delta(t - |\mathbf{x}|/c_0)}{4\pi|\mathbf{x}|c_0^2} \ ,$$

show that the far-field sound generated by a compact quadrupole distribution is

$$\rho'(\mathbf{x}, t) = \frac{x_i x_j \ddot{S}_{ij}(t - |\mathbf{x}|/c_0)}{4\pi c_0^4 |\mathbf{x}|^3} \qquad \text{where} \quad S_{ij}(t) = \int T_{ij}(\mathbf{y}, t)\mathrm{d}V \tag{2}$$

and $\dot{}$ denotes differentiation with respect to t.

Show further that $\rho' = O(m^4)$, where m is the fluctuation Mach number.

(ii) Suppose now that a moving body, surface $F(\mathbf{x}, t) = 0$, is introduced into the (inviscid) fluid, so that the right hand side of equation (1) has the additional terms

$$+\frac{\partial}{\partial t}(\rho_0 V_n \delta(F)) - \nabla.(p\mathbf{n}\delta(F)) \ ,$$

where ρ_0 is the mean density of the fluid, V_n is the body normal velocity, $\mathbf{n}$ is the (outward) body normal and p is the fluid pressure. Show that the expression in equation (2) for the far-field density fluctuation in the compact limit is augmented by

$$+\frac{\rho_0}{4\pi|\mathbf{x}|c_0^2}\dot{\mathcal{V}}(t - |\mathbf{x}|/c_0) + \frac{1}{4\pi|\mathbf{x}|^2 c_0^2}\dot{\mathcal{F}}(t - |\mathbf{x}|/c_0) \ ,$$

where

$$\mathcal{V}(t) = \int V_n(\mathbf{y}, t)\mathrm{d}S \qquad \mathcal{F}(t) = \int p(\mathbf{y}, t)\mathbf{x}.\mathbf{n}\mathrm{d}S$$

and the integrals are taken over the body surface.

(iii) A stationary air bubble in water executes small pulsations in such a way that its radius is $a_0 + a_1(t)$, where a_0 is a constant, $|a_1/a_0| \ll 1$ and $|\dot{a_1}/c_0| \ll 1$. By assuming that the fluid motion close to the bubble can be modelled as being strictly incompressible and irrotational, calculate explicitly the contributions to the total far-field sound represented by parts (i) and (ii) above.

Comment on the relative orders of magnitudes of the contributions from (i) and (ii).

[*Hint: Recall that* $\int \frac{x_i x_j}{r^6}\mathrm{d}V = \lambda\delta_{ij}$ *for some* λ*, to be determined, where the integration is completed over all* $r \geq a_0$*. Note also that in this limit* $T_{ij} \approx \rho_0 u_i u_j$*.*]

5 (a) Consider the equation

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + (1+x)\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0 \quad \text{with} \quad y(0) = y(1) = 1 \ .$$

Find the first two terms in the inner and outer expansions in the limit $\epsilon \to 0$. Calculate a uniformly-valid approximation to y which is correct up to and including $O(\epsilon)$.

Without further detailed calculation, **briefly** describe how you would attempt to determine the asymptotic solution of:

(i)

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - (1+x)\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0 \quad \text{with} \quad y(0) = y(1) = 1 \ .$$

(ii)

$$\epsilon \frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + x\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0 \quad \text{with} \quad y(0) = y(1) = 1 \ .$$

(b) Use the method of multiple scales to determine the small-ϵ approximation, valid for time $t = O(1/\epsilon)$, to the solution of the equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \epsilon \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^n + y = 0 \quad \text{with} \ \ y(0) = 1, \frac{\mathrm{d}y}{\mathrm{d}t(0)} = 0 \ ,$$

where n is a given positive integer. In your answer distinguish carefully between the cases of n even and n odd.

In the case of n large and odd, describe the behaviour of your solution for different orders of magnitude of t.