UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Thursday 29 May 2003 9 to 12

PAPER 73

SLOW VISCOUS FLOW

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Show that

$$\mathbf{u} = 2\boldsymbol{\Phi} - \nabla(\boldsymbol{\Phi}\cdot\mathbf{x}) + \nabla\chi \quad \text{with} \quad \nabla^2\boldsymbol{\Phi} = \nabla^2\chi = 0,$$

represents a possible Stokes flow in fluid of viscosity μ. Determine the corresponding pressure.

Hence or otherwise find the flow in unbounded fluid due to: (a) a point force $\mathbf{F}$; and (b) a point source of strength Q. Hence write down the flows generated by: (c) a stresslet; and (d) a source dipole.

Suppose now that viscous fluid occupies the region $\mathbf{x}\cdot\mathbf{n} > 0$ bounded by a rigid plane wall $\mathbf{x}\cdot\mathbf{n} = 0$ with $|\mathbf{n}| = 1$. A point force $F\mathbf{n}$ is placed at position $\mathbf{x} = d\mathbf{n}$ in the fluid. Show that the boundary conditions may be satisfied by placing at the image point $\mathbf{x} = -d\mathbf{n}$ an equal and opposite point force $-F\mathbf{n}$ along with a stresslet and a source dipole at the image point of strengths to be determined.

What is the far-field behaviour of $\mathbf{u}$?

2 Derive the boundary conditions that apply at an interface between two fluids having non-constant surface tension coefficient $\gamma(\mathbf{x})$.

The surface tension coefficient of a small spherical bubble of radius a depends on temperature T. The bubble is placed in unbounded fluid of constant viscosity μ and constant thermal diffusivity κ. Gravity is negligible. In the presence of a weak temperature gradient $\mathbf{H}$, and on the assumption that advection of heat is negligible, find a suitable Papkovich-Neuber potential and hence calculate the thermophoretic velocity of the bubble.

Identify carefully the dimensionless numbers that must be small for your analysis to be appropriate.

Explain physically what effect the presence of surfactants would have on your result.

3 A two-dimensional viscous drop, having uniform cross-section with area A, spreads in the x-direction under gravity g on a rigid horizontal surface. The kinematic viscosity of the fluid is ν. Use lubrication theory to obtain the evolution equation for the drop height $h(x,t)$ as

$$\frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{\partial}{\partial x}\left(h^3\frac{\partial h}{\partial x}\right).$$

Explain clearly the approximations made, and obtain the dimensionless groups that must be small for your analysis to be appropriate.

Obtain a similarity solution of this equation, and obtain in particular the drop width as a function of time.

[You may leave your answer in terms of $I = \int_0^{\pi/2} \cos^{5/3}\theta\, d\theta$.]

4 A Brownian particle of mass m is placed in viscous fluid so that it experiences a constant viscous resistance coefficient ζ. The particle position $x(t)$ satisfies the one-dimensional Langevin equation

$$m\ddot{x} + \zeta\dot{x} = f(t).$$

Suppose that $x = \dot{x} = 0$ at $t = 0$. By means of a Green's function or otherwise find $x(t)$ and hence $\dot{x}(t)$ for $t > 0$.

Suppose that $f(t)$ is a random function with zero mean and autocorrelation

$$< f(t)f(t+\tau) >= F\delta(\tau).$$

Suppose also that for $t \to \infty$,

$$\frac{1}{2}m < \dot{x}^2 > \to \frac{1}{2}kT.$$

Find F, and deduce $\lim_{t\to\infty} < x >$ and $\lim_{t\to\infty} < x^2 >$.

How is your result related to the diffusion equation

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial P}{\partial x}\right) \quad ?$$

Suppose instead that the fluid viscosity is a slowly-varying function of position so that

$$\zeta = \zeta(x) = \zeta_0 + \epsilon x \zeta_0' \quad \text{with} \quad \epsilon \ll 1.$$

By writing $x(t) = x_0(t) + \epsilon x_1(t)$, find $\lim_{t\to\infty} < x_1(t) >$ and deduce that the particle has a mean drift velocity to be found.

How can this result be derived from the diffusion equation?

Explain physically why the drift arises.