UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 9 June 2003 9 to 12

PAPER 72

FUNDAMENTALS OF ATMOSPHERE-OCEAN DYNAMICS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

Clarity and explicitness of reasoning will attract more credit than perfection of computational detail

(x, y, z) denotes right-handed Cartesian coordinates and (u, v, w) the corresponding velocity components; t is time; the gravitational acceleration is $(0, 0, -g)$ where g is a positive constant; $\hat{\mathbf{z}} = (0, 0, 1)$ is a unit vector directed vertically upward.
The fluid is always incompressible. 'Ideal fluid' always means that buoyancy diffusion can be neglected where relevant, as well as viscosity. N denotes the buoyancy frequency of a stratified fluid.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Write down the Boussinesq equations linearized about a state of rest with $N = N(z)$, for an ideal stably stratified, non-rotating fluid. Show that

$$\frac{\partial \mathcal{E}}{\partial t} + \nabla \cdot (p\mathbf{u}) \;=\; 0 \;, \tag{$*$}$$

where

$$\mathcal{E} \;=\; \tfrac{1}{2}|\mathbf{u}|^2 + \tfrac{1}{2}\frac{\sigma^2}{N^2} \;,$$

where $\mathbf{u}$ and σ denote the disturbance velocity and buoyancy acceleration, and p denotes the disturbance pressure divided by the constant inertial mass density ρ_{00}.

Starting from the same linearized equations, now taking N constant, and using appropriately tilted coordinate axes or otherwise, derive the dispersion relation for a plane progressive internal gravity wave. Verify that the same dispersion relation follows from the equation

$$\nabla^2 \zeta_{tt} + N^2(\zeta_{xx} + \zeta_{yy}) = 0 \;. \tag{$\dagger$}$$

(You need not derive this equation.) Here subscripts denote partial derivatives, ζ is the vertical component of the disturbance displacement field $\boldsymbol{\xi}$, and x, y, z are the standard coordinates with z vertical.

Show in a sketch how the $\mathbf{u}$, $\boldsymbol{\xi}$, σ, and p fields are distributed in space at a given instant, for a wave whose phase velocity is directed downward and leftward. Find a formula for the group velocity $\mathbf{c}_{\mathrm{g}}$ and show that $\overline{p\mathbf{u}} = \mathbf{c}_{\mathrm{g}}\overline{\mathcal{E}}$, where the overbars denote averages over a wavelength or period. Indicate the direction of $\mathbf{c}_{\mathrm{g}}$ in your sketch of disturbance fields, and explain this direction in terms of the pattern of positive and negative signs in the p distribution.

Two-dimensional internal gravity waves are generated by a moving boundary $z = a(T)\exp(ikx - i\omega t)$ (real part understood), where $T = \mu t$ and where μ, k, and ω are real positive constants, with $\mu \ll 1$ and $|\omega| < N$. The slowly-varying amplitude $a(T)$ is zero for $T \leqslant 0$ and increases smoothly to a constant, small positive value ϵ as T increases from 0 to 1, then remains at ϵ for $T \geqslant 1$. Taking ($\dagger$) as the starting point, show, correct to leading order in μ, that ζ takes the form

$$\zeta \;=\; a\left(T - \frac{Z}{\mathbf{c}_{\mathrm{g}} \cdot \hat{\mathbf{z}}}\right) \exp(ikx + imz - i\omega t) \;,$$

where $Z = \mu z$ and where m is a constant to be determined, with careful attention to its sign. Sketch the Z-dependence of the amplitude factor a at $T = 2$. Find the corresponding expressions for $\mathbf{u}$, σ, and p.

Show, **either** from your solution **or** from ($*$), that the vertical integral of $\overline{\mathcal{E}}$ increases at a rate equal to ${\rho_{00}}^{-1}$ times $\overline{W}$, the mean rate of working by the boundary on the fluid, per unit horizontal area.

By writing $\boldsymbol{\xi} = (\xi, 0, \zeta)$ and using the formula $\overline{u}_t = -\left[\,\overline{\xi_x\, u} + \overline{\zeta_x\, w}\,\right]_t + O(\mu^2)$, or otherwise, show that $\overline{W}$ is equal to minus the rate of change of mean-flow kinetic energy, per unit horizontal area, measured in the alternative frame of reference in which the boundary undulations are stationary.

2 Write down the fully nonlinear Boussinesq momentum and mass-continuity equations for two-dimensional motion ($\partial/\partial y = 0$) of an ideal stably stratified, non-rotating fluid. Derive the y-component of the corresponding vorticity equation in the form

$$\nabla^2\psi_t \;+\; \sigma_x \;=\; \psi_x\,\nabla^2\psi_z - \psi_z\,\nabla^2\psi_x \;,$$

where $\psi(x,z,t)$ is a streamfunction to be specified and σ is the departure of the buoyancy acceleration from its background value. Subscripts denote partial differentiation. Write down the fully nonlinear equation for σ, again using the streamfunction ψ.

Linearize the equations for $\nabla^2\psi$ and σ about a background state in which the buoyancy frequency N is a function of z, $N = N(z)$, and in which there is a steady flow in the positive x direction with velocity $\big(\bar{u}(z),\,0,\,0\big)$. Show that stationary wave disturbances of the form $\psi' = \widehat{\psi}(z)\,\exp(ikx)$ (real part understood) satisfy

$$\widehat{\psi}_{zz} \;+\; m^2(z)\,\widehat{\psi} \;=\; 0\;,$$

where

$$m^2(z) \;=\; \frac{N^2(z)}{\bar{u}^2(z)} - \frac{\bar{u}_{zz}}{\bar{u}} - k^2\;.$$

Find the general solution for $\widehat{\psi}$ in all cases in which the background state has exponential profiles $\bar{u}(z) = U_0\,\exp(Mz)$ and $N(z) = N_0\,\exp(Mz)$, where U_0, M, and N_0 are positive constants. Briefly discuss whether the Rayleigh quotient

$$\frac{\displaystyle\int_{z_1}^{z_2}\left\{\left(\frac{N^2(z)}{\bar{u}^2(z)} - \frac{\bar{u}_{zz}}{\bar{u}}\right)\widehat{\psi}^2 \;-\; \widehat{\psi}_z^2\right\}dz}{\displaystyle\int_{z_1}^{z_2}\widehat{\psi}^2 dz}$$

is well defined and equal to k^2 for any subset of these solutions for which $\widehat{\psi}_z = 0$ at $z = z_1$, taking care to specify any relevant restrictions on the values of U_0, M, N_0, and k.

Show that the full set of solutions has the property that $\nabla^2\psi \propto \psi$, with a constant of proportionality to be determined. Show that σ is *not* proportional to ψ except when $M = 0$. Discuss whether any of the solutions satisfy the fully nonlinear equations as well as the linearized equations.

3 Write down the shallow-water equations in a frame of reference rotating with constant angular velocity $(0, 0, \frac{1}{2}f)$, in the case of a gently undulating bottom boundary $z = b(x, y)$. Denote the layer depth by $h(x, y, t)$ with $h = h_{00} + \zeta - b$, so that $\zeta(x, y, t)$ represents the elevation of the free surface above its undisturbed level $z = h_{00}$. Derive the vertical component of the vorticity equation from the momentum equations, and show that Rossby's potential vorticity q_a/h is an exact material invariant, i.e. that

$$\frac{\mathcal{D}_H}{\mathcal{D}t}\left(\frac{q_a}{h}\right) = 0 \,, \qquad (*)$$

where $q_a = f + \partial v/\partial x - \partial u/\partial y$ and where $\mathcal{D}_H/\mathcal{D}t = \partial/\partial t + \mathbf{u}_H \cdot \nabla_H$, with $\mathbf{u}_H = (u, v, 0)$ and $\nabla_H = (\partial/\partial x,\, \partial/\partial y,\, 0)$.

Using order-of-magnitude arguments and making appropriate assumptions about small parameters, carefully derive the quasi-geostrophic counterpart to $(*)$, expressing everything in terms of an appropriate streamfunction $\psi(x, y, t)$.

Find ψ for the case of uniform steady flow $\mathbf{u}_H = (U,\, 0,\, 0)$, where U is a positive constant, and show that if $b = -g^{-1}fUy$ then h is constant. A low ridge

$$\hat{b} = \hat{b}(x) = \begin{cases} \epsilon \cos \dfrac{x}{L} \,, & |x| < \frac{1}{2}\pi L \\ 0 \,, & |x| > \frac{1}{2}\pi L \end{cases}$$

is added $(b = -g^{-1}fUy + \hat{b})$, where L is a positive constant, and $\epsilon \ll h_{00}$. Show that ψ changes by a function $\hat{\psi}(x)$ satisfying

$$\frac{d^2\hat{\psi}}{dx^2} - L_R^{-2}\hat{\psi} = -\frac{f}{h_{00}}\,\hat{b}(x)\,, \quad \text{with } \hat{\psi} \to 0 \text{ as } |x| \to \infty\,,$$

where L_R is the Rossby length based on h_{00}. (Continue to use quasi-geostrophic theory and assume that the flow is steady, with $\mathbf{u}_H = (U,\, 0,\, 0)$ far upstream.) Solve for $\hat{\psi}$ and find a formula for the shape $y = y(x)$ of a typical streamline. Show this shape on a sketch of the xy plane in the case $L = 3L_R$.

Briefly discuss how the steady quasi-geostrophic flow might be set up from an initial state in which $\mathbf{u}_H = (u(x), 0, 0)$, with $h = h_{00}$ far upstream and with

$$u(x) = \begin{cases} U\left\{1 - h_{00}^{-1}\epsilon \cos(x/L)\right\}^{-1} \,, & |x| < \frac{1}{2}\pi L \\ U \,, & |x| > \frac{1}{2}\pi L \,. \end{cases}$$

Include some brief mention of how the initial Rossby potential-vorticity distribution might evolve, and of the way in which the flow might approach the state of geostrophic balance assumed by the quasi-geostrophic solution.

4 Write an essay on the concept of Rossby-wave propagation. You should discuss and illustrate the generic properties common to all cases, internal and boundary-dependent, such as the relation to potential-vorticity gradients, the scale dependence arising from potential-vorticity inversion, and the lack of two-way phase propagation as contrasted with other wave types. You should also mention the relevant parameter regimes and orders of magnitude. Further aspects to bring in might include, for instance, (*a*) the relation to the short-wave limit of the Eady dispersion relation, and (*b*) the simplest model of potential-vorticity mixing due to Rossby-wave breaking, with its implications for momentum transport.

[*The Eady dispersion relation, relating complex phase speed* c *to horizontal wavenumber* $|\mathbf{k}|$, *is* $c = \frac{1}{2}\Lambda H\big\{1 \pm \gamma^{-1}\big[(\gamma - \coth\gamma)(\gamma - \tanh\gamma)\big]^{1/2}\big\}$, *where* Λ *is the vertical shear of the basic flow,* H *is the vertical distance between the boundaries, and* $\gamma = \frac{1}{2}KH$ *where* K *is the inverse Rossby height based on* $|\mathbf{k}|$.]