UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2003 9 to 11

PAPER 70

THEORY OF ELASTIC SOLIDS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 (a) State and prove the polar decomposition theorem for a 3×3 matrix with positive determinant.

(b) Show that the length of a material fibre, in the direction of the referential unit vector $\mathbf{l}$, is not changed under the uniform deformation A provided that

$$\mathbf{l}^T A^T A \, \mathbf{l} = 1.$$

(c) A body is reinforced by embedding in it two families of inextensible fibres. The body has an undeformed configuration in which the fibres of each family are straight and parallel, and unit vectors defining their orientations have components $(\cos\theta, \sin\theta, 0)^T$, $0 \leqslant \theta \leqslant \pi/2$, relative to an orthonormal basis. The body is subjected to homogeneous pure stretch $A = U = \mathrm{diag}(\lambda^{-1/2}\alpha, \lambda^{-1/2}\alpha^{-1}, \lambda)$. Prove that

$$\alpha^2 = \{\lambda \pm (\lambda^2 - \sin^2 2\theta)^{1/2}\}/(2\cos^2\theta),$$

and deduce that

(i) when $1 \neq \lambda > \sin 2\theta$, two deformed configurations are possible,

(ii) when the maximum contraction in the 3-direction is achieved, the two families of fibres are orthogonal in the deformed configuration.

2 Given that for elastic material with stored energy function $W(A)$ per unit reference volume, the rate of working $N_{\alpha i}\dot{A}_{i\alpha}$ of the nominal stress must equal the rate of change of W during any allowed deformation, prove that, if the material is subject to a constraint $F(A) = 0$, then

$$N_{\alpha i} = \frac{\partial W}{\partial A_{i\alpha}} + q\frac{\partial F}{\partial A_{i\alpha}}$$

for some scalar q. Show in particular that, if the material is incompressible, then

$$N_{\alpha i} = \frac{\partial W}{\partial A_{i\alpha}} + q(A^{-1})_{\alpha i}. \qquad (*)$$

[*You are reminded that* $\det A = (1/6)\varepsilon_{ijk}\varepsilon_{\alpha\beta\gamma}A_{i\alpha}A_{j\beta}A_{k\gamma}$.]

A uniform plane sheet of incompressible, isotropic elastic material is subjected to equibiaxial tension, giving rise to equibiaxial extension λ in its plane. Given that W is expressible as a function of the principal stretches $(\lambda_1, \lambda_2, \lambda_3)$, show by use of $(*)$ that the in-plane nominal tension is given by

$$N_{11} = N_{22} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\lambda}W(\lambda, \lambda, \lambda^{-1/2}).$$

Explain why this result is consistent with energy conservation.

Now consider a spherical balloon of the same material, of initial internal radius a_0 and thickness $h_0 \ll a_0$. Prove by considering energy conservation that the internal pressure (force per current area) $p(a)$, when the balloon is inflated to radius a, is, to leading order in h_0/a_0,

$$p(a) = \left(\frac{h_0}{a_0}\right)\frac{1}{\lambda^2}\frac{\mathrm{d}}{\mathrm{d}\lambda}W(\lambda, \lambda, \lambda^{-2}),$$

where $\lambda = a/a_0$.

3 A body composed of neo-Hookean, incompressible elastic material with energy function $W(A) = (1/2)\mu A_{i\alpha}A_{i\alpha}$ occupies, in its reference configuration, the cylindrical region

$$\{(\xi_1, \xi_2, \xi_3) : \ \xi_1^2 + \xi_2^2 < R^2, \ 0 < \xi_3 < H\}.$$

It is subject to the combined stretching and torsional deformation

$$x_1 = \lambda^{-1/2}(\xi_1\cos(\alpha\xi_3) - \xi_2\sin(\alpha\xi_3)), \ x_2 = \lambda^{-1/2}(\xi_1\sin(\alpha\xi_3) + \xi_2\cos(\alpha\xi_3)), \ x_3 = \lambda\xi_3,$$

its curved surface remaining traction-free. Calculate the components of nominal traction on the end $\xi_3 = H$ and deduce that they exert a moment $\mu\pi R^4\alpha/(2\lambda)$ about the 3-axis.

4 (a) Isotropic elastic material with shear modulus μ and density ρ is subjected to a state of infinitesimal anti-plane strain, so that $\mathbf{u} = (0, 0, w(x_1, x_2))$. Write down the components of stress and deduce from the equations of motion that w satisfies the wave equation

$$\nabla^2 w - \frac{1}{c^2} w_{,tt}$$

in the absence of body force, where $c^2 = \mu/\rho$.

Suppose further that w depends on x_1 and t only in the combination $x = x_1 - Vt$. Deduce that w is expressible as the real part of an analytic function of $z = x + \mathrm{i}y$, where $y = (1 - V^2/c^2)^{1/2} x_2$.

(b) A plane crack in an infinite isotropic elastic medium propagates with uniform speed $V < c$ so that, at time t, it occupies the surface

$$S(t) = \{(x_1, x_2, x_3) : \; -\infty < x_1 - Vt < 0, \; x_2 = 0, \; -\infty < x_3 < \infty\}.$$

Its faces are loaded so that

$$\sigma_{32} \to -f(x_1 - Vt) \text{ as } x_2 \to \pm 0 \text{ on } S(t),$$

to induce a state of antiplane deformation. No forces are applied elsewhere.

Show that

$$(1 - V^2/c^2)^{1/2} \sigma_{13} - \mathrm{i}\sigma_{23} = \mu F'(z); \;\; z = x + \mathrm{i}(1 - V^2/c^2)^{1/2} x_2,$$

giving explicitly the function $F'(z)$. Deduce, in particular, that, directly ahead of the crack,

$$\sigma_{23}(x_1, 0, t) = \frac{1}{\pi x^{1/2}} \int_{-\infty}^{0} \frac{(-s)^{1/2} f(s) \mathrm{d}s}{x - s}; \;\; x = x_1 - Vt,$$

independently of the speed V.

[*You are reminded that, if $G(z)$ is analytic except on the segment $[a, b]$ of the real axis, and*

$$G(x + 0\mathrm{i}) - G(x - 0\mathrm{i}) = g(x), \;\; a \leqslant x \leqslant b,$$

then

$$G(z) = \frac{1}{2\pi \mathrm{i}} \int_a^b \frac{g(s) \mathrm{d}s}{s - z} + P(z),$$

where $P(z)$ is an entire function.]