UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Tuesday 3 June 2003 1.30 to 4.30

PAPER 7

RESTRICTION AND KAKEYA PHENOMENA

Attempt **ONE** *question from each of sections A, B and C*

There are **six** *questions in total.*

The questions carry equal weight.

The notation $o(1)$ refers to a quantity which tends to zero as some other quantity (which will always be obvious from the context) tends to infinity.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

Section A

1 Let $E \subseteq \mathbb{R}^n$. Define the *upper* and *lower Minkowski dimensions* of E. Let Q be the set of all real numbers between 0 and 1 whose base 9 expansions contain only the digits $2, 5$ and 7. Show that the upper and lower Minkowski dimensions of Q are both $1/2$. What is meant by the term *Besicovitch set*? Show that any Besicovitch set in $\mathbb{R}^2$ has upper and lower Minkowski dimension 2.

2 Let p be a prime. What is meant by a Besicovitch subset of $\mathbb{F}_p^n$? Show that there is a Besicovitch subset of $\mathbb{F}_p^2$ with cardinality $\frac{1}{2}p^2\,(1+o(1))$. Show that there is an absolute constant $c > 0$ so that every Besicovitch subset of $\mathbb{F}_p^{15}$ has cardinality at least cp^9.

Section B

3 Show that there is an absolute constant C with the following property. If $f : S^1 \to \mathbb{C}$ is any measurable function with $\|f\|_\infty = 1$, and if $R \geq 2$, then

$$\|\widehat{f d\sigma}\|_{L^4(B(0,R))} \;\leq\; C(\log R)^{1/4}.$$

(Here σ refers to the usual measure on the circle S^1, and $B(0,R)$ is the ball of radius R about the origin in $\mathbb{R}^2$). Sketch how such a result can be used to prove that any Besicovitch set in $\mathbb{R}^2$ has upper and lower Minkowski dimension 2 (details are not required).

4 Let p be a prime of the form $4k+3$, so that -1 is not a square in $\mathbb{F}_p$. What is meant by the *discrete paraboloid* $P \subseteq \mathbb{F}_p^3$? Determine $\widehat{d\sigma}(x_1, x_2, x_3)$, where σ is the normalised counting measure on P. Define $R^*(2 \to 4)$, and show that it is at most 10 (you may state and use, without proof, any facts concerning discrete functional analysis and discrete Fourier analysis that you may require). Now suppose that p is of the form $4k+1$, so that -1 is a square in $\mathbb{F}_p$. Show that P contains a line. Hence, or otherwise, show that $R^*(2 \to q)$ is not bounded as $p \to \infty$ for any $q < 4$.

Section C

5 (i) Prove that there is a quite fair boolean function $f : \mathbb{F}_2^n \to \{0,1\}$, all of whose influences are at most $10 \log n/n$. You should define all of the terms used in this part of the question.

(ii) State Beckner's inequality. Let $A \subseteq \mathbb{F}_2^n$ have cardinality $\lfloor N/2003 \rfloor$, where $N = 2^n$. Show that if $N > N_0$ is sufficiently large then

$$\int_{\xi : |\xi| \leq 2} \widehat{A}(\xi)^2 \, d\xi \;<\; \int_{\xi : |\xi| > 2} \widehat{A}(\xi)^2 \, d\xi \,.$$

6 (i) Let $A \subseteq \{1, \ldots, N\}$. We say that A is a *Sidon set* if the only solutions to the equation $a - b = c - d$ with $a, b, c, d \in A$ are those for which $a = b, c = d$, or $a = c, b = d$. Show that there is a Sidon subset of $\{1, \ldots, N\}$ with cardinality $\sqrt{N} - o(\sqrt{N})$ (you may assume that for sufficiently large positive real numbers x there is a prime p satisfying $x - x^{3/4} \leq p \leq x$).

(ii) Show that the largest Sidon subset $A \subseteq \{1, \ldots, N\}$ has size $\sqrt{N} + o(\sqrt{N})$ (Hints: write $u = \lfloor N^{3/4} \rfloor$, and define the numbers $A_i = |A \cap \{i - u + 1, \ldots, i\}|$ $(i = 1, \ldots, N + u)$. Estimate the quantity $\sum_i \frac{1}{2} A_i(A_i - 1)$ in two ways; using the Cauchy-Schwarz inequality, and by using the fact that A has the Sidon property).