UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 30 May 2003 1.30 to 4.30

# PAPER 67

# NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

*Attempt* **THREE** *questions from Section A and attempt* **ONE** *question from Section B.*

*Each question from Section B carries twice the weight of a question from Section A.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**Section A**

**1** The implicit midpoint rule for the ODE system $\mathbf{y}' = \mathbf{f}(t,\mathbf{y})$, $t \geqslant 0$, $\mathbf{y}(0) = \mathbf{y}_0$, is

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\mathbf{f}(t_n + \tfrac{1}{2}h, \tfrac{1}{2}(\mathbf{y}_n + \mathbf{y}_{n+1})), \qquad n \geqslant 0.$$

a. Formulate the implicit midpoint rule as a Runge–Kutta method and determine its order. Is the method A-stable?

b. Suppose that it is known that, for every initial condition $\mathbf{y}_0$, the solution of the ODE posesses the invariant $\mathbf{y}^\top(t)S\mathbf{y}(t) \equiv \mathbf{y}_0^\top S\mathbf{y}_0$, $t \geqslant 0$, where $S$ is a given symmetric matrix. Prove that $\mathbf{y}_n^\top S\mathbf{y}_n \equiv \mathbf{y}_0^\top S\mathbf{y}_0$, $n \geqslant 0$.

**2** Consider the two-step ODE method

$$\mathbf{y}_{n+2} - \tfrac{4}{2+\alpha}\mathbf{y}_{n+1} + \tfrac{2-\alpha}{2+\alpha}\mathbf{y}_n = \tfrac{h}{2+\alpha}(\mathbf{f}_{n+2} + 2\alpha\mathbf{f}_{n+1} - \mathbf{f}_n),$$

where $\mathbf{f}_m = \mathbf{f}(t_m, \mathbf{y}_m)$, while $\alpha \neq -2$ is a parameter.

a. Determine the range of $\alpha$ for which the method is convergent. For every such $\alpha$ compute the order of the method.

b. Prove that no $\alpha$ exists so that the method is both convergent and A-stable.

**3** The diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(a(x)\frac{\partial u}{\partial x}\right),$$

where $a$ is a positive function, is given for $0 \leqslant x \leqslant 1$, $t \geqslant 0$, with initial conditions at $t = 0$ and zero Dirichlet boundary conditions at $x = 0, 1$. It is solved by the fully-discretized finite difference method

$$u_{m+1}^{n+1} = u_m^n + \mu[a_{m-1/2}u_{m-1}^n - (a_{m-1/2} + a_{m+1/2})u_m^n + a_{m+1/2}u_{m+1}^n],$$

where $\mu = \Delta t/(\Delta x)^2$ and $a_\gamma = a(\gamma\Delta x)$.

a. Derive the order of magnitude of the local error.

b. Determine the range of $\mu > 0$ for which the method is stable for every function $a$ such that $0 < a_- \leqslant a(x) \leqslant a_+ < \infty$.

**4** The differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}, \qquad -\infty < x, y < \infty, \;\; t \geqslant 0,$$

specified with $L_2[\mathrm{R}^2]$ initial conditions at $t = 0$, is solved by the finite difference two-step method

$$u_{k,m}^{n+1} = \tfrac{\Delta t}{\Delta x}(u_{k+1,m}^n + u_{k,m+1}^n - u_{k-1,m}^n - u_{k,m-1}^n) + u_{k,m}^{n-1},$$

where $u_{k,m}^n \approx u(k\Delta x, m\Delta x, n\Delta t)$.

a. Determine the range of Courant numbers $\Delta t/\Delta x$ for which the method is stable.

b. Determine the range of Courant numbers $\Delta t/\Delta x$ for which the method converges.

**5** Consider the linear algebraic system of equations

$$\begin{aligned} -2x_1 + x_2 + x_n &= b_1, \\ x_{k-1} - 2x_k + x_{k+1} &= b_k, \qquad k = 2, 3, \ldots, n-1, \\ x_1 + x_{n-1} - 2x_n &= b_n, \end{aligned}$$

whose matrix is a circulant.

a. Write down explicitly the Jacobi and Gauss–Seidel iterative methods for this system.

b. Either prove or disprove the convergence of both iterative methods in the present case.

## Section B

**6** Write an essay on the design of finite-difference methods for the Poisson equation, inclusive of the *Mehrstellenverfahren* technique.

**7** Write an essay on the multigrid method. You should provide a justification for the method, describe in detail the implementation of a V-cycle and comment upon the performance of the method.