UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 6 June 2003 1.30 to 4.30

PAPER 65

ASTROPHYSICAL DISCS AND MAGNETOHYDRODYNAMICS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A gas flows according the equations of compressible, ideal magnetohydrodynamics

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\rho \nabla \Phi - \nabla p + \frac{1}{\mu_0} (\nabla \times \mathbf{B}) \times \mathbf{B},$$

$$\rho T \left(\frac{\partial s}{\partial t} + \mathbf{u} \cdot \nabla s \right) = 0,$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}),$$

$$\nabla \cdot \mathbf{B} = 0.$$

The flow is such that all physical quantities are independent of t and z, when referred to Cartesian coordinates (x, y, z).

(i) Argue that the velocity and magnetic field are related by

$$\mathbf{u} = \frac{k \mathbf{B}}{\rho} + U \, \mathbf{e}_z,$$

where k and U are constant on each magnetic field line.

(ii) Show further that s, and also the quantity

$$V = u_z - \frac{B_z}{\mu_0 k},$$

are constant on each magnetic field line. Deduce that, unless B_z vanishes identically on a magnetic field line, the horizontal velocity and the horizontal Alfvén velocity cannot coincide at any single point of the field line.

(iii) Consider the quantity

$$Q = \frac{1}{2} u^2 + \Phi + w - \frac{U B_z}{\mu_0 k},$$

where w is the specific enthalpy, such that $\mathrm{d}w = T \, \mathrm{d}s + \rho^{-1} \, \mathrm{d}p$. Show that Q is constant on each magnetic field line, and discuss the physical interpretation of this conservation law.

2 (i) Starting from the equation of mass conservation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0,$$

and the equation of motion in the form

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\rho \nabla \Phi - \nabla p + \nabla \cdot \mathbf{T},$$

derive the diffusion equation

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r} \frac{\partial}{\partial r} \left[r^{1/2} \frac{\partial}{\partial r} \left(r^{1/2} \bar{\nu} \Sigma \right) \right] \qquad (*)$$

governing the surface density $\Sigma(r,t)$ of a Keplerian disc. You should state any approximations or assumptions you require, and explain how the mean kinematic viscosity $\bar{\nu}(r,t)$ is related to the stress tensor $\mathbf{T}$.

[You may assume that the divergence of a vector field $\mathbf{A}$ and a symmetric second-rank tensor field $\mathbf{B}$ in cylindrical polar coordinates (r, ϕ, z) are

$$\nabla \cdot \mathbf{A} = \frac{1}{r} \frac{\partial}{\partial r}(r A_r) + \frac{1}{r} \frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z},$$

$$\begin{aligned} \nabla \cdot \mathbf{B} = & \left[\frac{1}{r} \frac{\partial}{\partial r}(r B_{rr}) + \frac{1}{r} \frac{\partial B_{r\phi}}{\partial \phi} + \frac{\partial B_{rz}}{\partial z} - \frac{B_{\phi\phi}}{r} \right] \mathbf{e}_r \\ & + \left[\frac{1}{r^2} \frac{\partial}{\partial r}(r^2 B_{r\phi}) + \frac{1}{r} \frac{\partial B_{\phi\phi}}{\partial \phi} + \frac{\partial B_{\phi z}}{\partial z} \right] \mathbf{e}_\phi \\ & + \left[\frac{1}{r} \frac{\partial}{\partial r}(r B_{rz}) + \frac{1}{r} \frac{\partial B_{\phi z}}{\partial \phi} + \frac{\partial B_{zz}}{\partial z} \right] \mathbf{e}_z .] \end{aligned}$$

(ii) Consider a disc with the viscosity law

$$\bar{\nu} = A r^2 \Sigma,$$

where A is a constant. Show that solutions of the form

$$\Sigma = \sigma(t) \left\{ \left[\frac{R(t)}{r} \right]^a - 1 \right\}, \qquad r \leq R(t),$$

exist for only two non-zero values of the parameter a, namely $a = 1$ and $a = 5/4$. In each case, solve for $\sigma(t)$ and $R(t)$, assuming that $R(0) = 0$.

(iii) For the solution with $a = 1$, show that the mean radial velocity in the disc is

$$\bar{u}_r = -\frac{(R - 5r)}{10t}.$$

Determine the trajectories of fluid elements moving with the mean radial velocity, and deduce that almost every fluid element is accreted in a finite time. For the solution with $a = 5/4$, show that $\bar{u}_r$ is strictly positive.

(iv) Investigate whether mass and angular momentum are globally conserved in either of the two solutions. Comment on the likely significance of these special solutions among solutions of equation $(*)$ as an initial-value problem with various initial and boundary conditions.

3 Let (r, ϕ, z) be cylindrical polar coordinates, and let $\Phi(r, z)$ be an axisymmetric gravitational potential that is symmetric about the mid-plane $z = 0$.

(i) Show that the angular velocity $\Omega(r)$, epicyclic oscillation frequency $\kappa(r)$ and vertical oscillation frequency $\Omega_z(r)$ associated with circular test-particle orbits in the mid-plane are given by

$$\Omega^2 = \frac{1}{r}\Phi_{,r}(r, 0),$$

$$\kappa^2 = \Phi_{,rr}(r, 0) + \frac{3}{r}\Phi_{,r}(r, 0),$$

$$\Omega_z^2 = \Phi_{,zz}(r, 0),$$

where the subscript comma denotes partial differentiation.

(ii) Accretion on to a non-rotating black hole can be modelled using Newtonian dynamics, by using a gravitational potential that differs from that of a point mass. Consider the potential

$$\Phi = -\frac{GM}{(r^2 + z^2)^{1/2} - a}, \qquad r^2 + z^2 > a^2,$$

where a represents the radius of the black hole. Find $\Omega(r)$, $\kappa(r)$ and $\Omega_z(r)$ for this potential, and deduce that circular test-particle orbits are unstable for $a < r < 3a$.

(iii) A particle, initially in a circular orbit of radius $3a$, is given an infinitesimal inward radial displacement. Show that it spirals into the black hole, and that its velocity at radius r, for $a < r < 3a$, is

$$\mathbf{v} = -\left[\frac{GM(3a - r)^3}{4ar^2(r - a)}\right]^{1/2} \mathbf{e}_r + \left[\frac{27GMa}{4r^2}\right]^{1/2} \mathbf{e}_\phi.$$

(iv) Consider a steady, thin accretion disc around the black hole. You may assume that the disc is composed of an ideal gas, is strictly isothermal, and has an effective viscosity given by the alpha prescription. Show that the fluid velocity field $\mathbf{u} = \mathbf{v}$, where $\mathbf{v}$ is as given above, provides a good approximate solution of the equation of motion of the fluid in the region $a < r < 3a$, within this Newtonian model. Estimate the order of magnitude of the fractional error in this approximation, in terms of the characteristic angular semi-thickness (H/r) of the disc. Argue that the viscous torque at $r = 2a$, say, is smaller than that at $r = 4a$, say, by a factor of order $\alpha(H/r)^2$.

4 The *shearing box* is a local model of a differentially rotating disc. The velocity perturbation $\mathbf{v}$ and magnetic field $\mathbf{B}$ in an incompressible shearing box of uniform density ρ, kinematic viscosity ν and magnetic diffusivity η satisfy the same equations as in the shearing sheet,

$$\left(\frac{\partial}{\partial t} - 2Ax\frac{\partial}{\partial y}\right)\mathbf{v} - 2Av_x\,\mathbf{e}_y + 2\Omega\,\mathbf{e}_z \times \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} = -\nabla\psi + \frac{1}{\mu_0\rho}\mathbf{B}\cdot\nabla\mathbf{B} + \nu\nabla^2\mathbf{v},$$

$$\left(\frac{\partial}{\partial t} - 2Ax\frac{\partial}{\partial y}\right)\mathbf{B} + 2AB_x\,\mathbf{e}_y + \mathbf{v}\cdot\nabla\mathbf{B} = \mathbf{B}\cdot\nabla\mathbf{v} + \eta\nabla^2\mathbf{B},$$

$$\nabla\cdot\mathbf{v} = \nabla\cdot\mathbf{B} = 0.$$

However, unlike the shearing sheet, these equations are solved in a finite box

$$0 < x < L_x, \qquad 0 < y < L_y, \qquad 0 < z < L_z.$$

If f denotes ψ or any component of $\mathbf{v}$ or $\mathbf{B}$, the boundary conditions are

$$\begin{aligned} f(L_x,y,z,t) &= f(0,y',z,t),\\ f(x,L_y,z,t) &= f(x,0,z,t),\\ f(x,y,L_z,t) &= f(x,y,0,t), \end{aligned} \qquad (**)$$

where

$$y' = (y + 2AL_xt) \bmod L_y.$$

(i) Explain briefly how the quantities Ω, A and ψ in the shearing sheet relate to the physical quantities in a differentially rotating disc.

(ii) Let angle brackets denote an average over the volume of the box, i.e.

$$\langle f\rangle(t) = \frac{1}{L_xL_yL_z}\int_0^{L_z}\int_0^{L_y}\int_0^{L_x} f(x,y,z,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z.$$

If $Q(x,y,z,t)$ denotes any quantity satisfying the boundary conditions $(**)$, show that

$$\left\langle\frac{\partial Q}{\partial x}\right\rangle = \left\langle\frac{\partial Q}{\partial y}\right\rangle = \left\langle x\frac{\partial Q}{\partial y}\right\rangle = \left\langle\frac{\partial Q}{\partial z}\right\rangle = 0.$$

(iii) Derive the volume-averaged equations

$$\frac{\mathrm{d}\langle\mathbf{v}\rangle}{\mathrm{d}t} - 2A\langle v_x\rangle\,\mathbf{e}_y + 2\Omega\,\mathbf{e}_z\times\langle\mathbf{v}\rangle = \mathbf{0},$$

$$\frac{\mathrm{d}\langle\mathbf{B}\rangle}{\mathrm{d}t} + 2A\langle B_x\rangle\,\mathbf{e}_y = \mathbf{0}.$$

Deduce that the box as a whole can undergo epicyclic oscillations. Explain why it is impossible for an accretion flow to develop in the shearing box.

(iv) Derive the energy-like equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \frac{1}{2}v^2 + \frac{B^2}{2\mu_0\rho} \right\rangle = 2A\left\langle v_x v_y - \frac{B_x B_y}{\mu_0\rho} \right\rangle - \nu\left\langle |\nabla \times \mathbf{v}|^2 \right\rangle - \frac{\eta}{\mu_0\rho}\left\langle |\nabla \times \mathbf{B}|^2 \right\rangle.$$

Let $L = \max(L_x, L_y, L_z)$, and define the magnetic Reynolds number

$$\mathrm{Rm} = \frac{L^2|A|}{\eta}.$$

Show that, if $\nu = \eta$ and $\langle \mathbf{v} \rangle = \langle \mathbf{B} \rangle = \mathbf{0}$, turbulence cannot be sustained in the box if $\mathrm{Rm} < 4\pi^2$.