UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 6 June 2003 9 to 12

PAPER 64

ASTROPHYSICAL FLUID DYNAMICS

Attempt **TWO** *questions.*

There are **three** *questions in total.*

The questions carry equal weight.

Candidates may bring their notebooks into the examination. The following equations may be assumed.

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{u} = 0$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p - \rho \nabla \Phi + \mathbf{j} \wedge \mathbf{B}$$

$$\rho \frac{De}{Dt} = \frac{p}{\rho} \frac{D\rho}{Dt} + \operatorname{div}(\lambda \nabla T) + \epsilon$$

$$\operatorname{div} \mathbf{B} = 0; \;\; \mathbf{j} = \mu_o^{-1} \operatorname{curl} \mathbf{B}$$

$$\nabla^2 \Phi = 4\pi G \rho$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{u} \wedge \mathbf{B})$$

.

$$p = (\gamma - 1)\rho e = \frac{\mathcal{R}}{\mu} \rho T$$

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 A large spherical cloud of gas, radius R_o, is centred at the origin, and has uniform density ρ_{oo}, and zero pressure. At time $t = 0$ it begins to collapse from rest under its own gravity. At time t the velocity field within the cloud may be written

$$\mathbf{u}_o(\mathbf{r}, t) = \mathbf{r}\left[\frac{\dot{R}(t)}{R(t)}\right],$$

where $R(t)$ is the cloud radius and $\mathbf{r}$ the radius vector from the origin. Show that the density remains uniform, and that at time t the density is

$$\rho_o(t) = \rho_{oo}\left[R_o/R(t)\right]^3.$$

Show that the gravitational force within the cloud is

$$\mathbf{g} = -\mathbf{r}\left[\frac{4}{3}\pi G\rho_o(t)\right],$$

and hence that

$$R^2\ddot{R} = -\frac{4}{3}\pi G\rho_{oo}R_o^3.$$

Show that the collapse is described implicitly by

$$R = \frac{1}{2}R_o(1 + \cos\phi),$$

and

$$Ct = \frac{1}{2}R_o(\phi + \sin\phi),$$

where

$$C^2 = 8\pi G\rho_{oo}R_o^2/3.$$

As the cloud collapses (still with zero pressure) it is subject to small perturbations. Ignoring the effects of the cloud boundaries (or, assuming the cloud is infinitely large) we assume that the perturbations take the form

$$\rho(\mathbf{r}, t) = \rho_o(t) + \rho'(\mathbf{r}, t),$$

with

$$\rho'(\mathbf{r}, t) = \rho_1(t)\exp(i\mathbf{k}.\mathbf{r}),$$

and where

$\mathbf{k}(t) = \mathbf{q}/R(t)$, and $\mathbf{q}$ is independent of t. Explain the physical motivation for such an assumption.

Using this assumption show that

$$\dot{\rho}_1 + \frac{3\dot{R}}{R}\rho_1 + \frac{i\rho_o}{R}\mathbf{q}.\mathbf{u}_1 = 0,$$

where $\mathbf{u_1}(t)$ is the analogous velocity perturbation.

Hence, or otherwise, show that for compressive modes the fractional density perturbation $\delta(t) \equiv \rho_1(t)/\rho_o(t)$ satisfies the equation

$$\ddot{\delta} + \frac{2\dot{R}}{R}\dot{\delta} - 4\pi G\rho_o(t)\delta = 0.$$

Show that one solution of this equation is given (implicitly) by

$$\delta = \frac{\sin\phi}{(1+\cos\phi)^2} \ .$$

By considering the behaviour of the solutions as $R \to 0$ or otherwise, show that this is the only growing solution as the collapse proceeds.

2 Show that the magnetohydrodynamic equations for a non-viscous, infinitely conducting fluid which obeys the perfect gas law may be written in the form

$$\frac{\partial \rho}{\partial t} + \text{div }(\rho \mathbf{u}) = 0,$$

$$\frac{\partial}{\partial t}(\rho \mathbf{u}) + \text{div }(\rho \mathbf{u}\mathbf{u} + pI - T) = -\rho \nabla \Phi,$$

and

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho u^2 + \rho e + \rho \Phi + \frac{1}{2\mu_o}B^2\right)$$

$$+ \text{ div }\left[\rho \mathbf{u}\left(e + \frac{p}{\rho} + \frac{1}{2}u^2 + \Phi\right) + \mu_o^{-1}\left(\mathbf{B}\wedge\mathbf{u}\right)\wedge\mathbf{B}\right] = \rho\frac{\partial \Phi}{\partial t},$$

where I is the unit tensor and the tensor T is given by

$$T = \mu_o^{-1}\left(\mathbf{B}\mathbf{B} - \frac{1}{2}B^2 I\right).$$

Show that in a steady hydromagnetic shock in which the magnetic field and the flow velocity are normal to the shock front, the jump conditions across the shock front are the usual Rankine-Hugoniot conditions. Give a brief physical explanation of why the magnetic field plays no role in this case.

Now consider the jump conditions for a hydromagnetic shock in which the magnetic field is parallel to the shock front and the flow velocity is normal to it. Show that $B_1/\rho_1 = B_2/\rho_2$, where the subscripts 1, 2 refer to pre- and post-shock quantities, respectively.

Show that the jump conditions concerning conservation of mass and momentum imply that

$$p_2 = p_1 + \rho_1 u_1^2\left(1 - \frac{\rho_1}{\rho_2}\right) + \frac{B_1^2}{2\mu_o}\left(1 - \frac{\rho_2^2}{\rho_1^2}\right).$$

Use the jump condition concerning energy conservation to obtain another expression for p_2 in terms of pre-shock variables.

Hence, or otherwise show that

$$(2-\gamma)v_{A1}^2 x^2 + \left[(\gamma-1)u_1^2 + 2c_1^2 + \gamma v_{A1}^2\right]x - (\gamma+1)\,u_1^2 = 0,$$

where $x = \rho_2/\rho_1$, $c_1^2 = \gamma p_1/\rho_1$ and $v_{A1}^2 = B_1^2/\mu_o\rho_1$.

Deduce that for the shock to exist we require that

$$v_1^2 > c_1^2 + v_{A1}^2,$$

and give a physical interpretation of this condition.

[You may assume that div $(\mathbf{a}\wedge\mathbf{b}) = \mathbf{b}\,.\,\text{curl }\mathbf{a} - \mathbf{a}\,.\,\text{curl }\mathbf{b}$.]

3 An incompressible fluid of uniform density ρ has a uniform magnetic field $\mathbf{B}_o = (B, 0, 0)$, in Cartesian coordinates, and a shearing velocity field $\mathbf{u}_o = (U(z), 0, 0)$.

The flow is subject to small perturbations of the form

$$\mathbf{u} = \mathbf{u}_o + [u(z), v(z), w(z)] \exp\left(i\sigma t + ik_x x + ik_y y\right),$$

$$\mathbf{B} = \mathbf{B}_o + [b_x(z), b_y(z), b_z(z)] \exp\left(i\sigma t + ik_x x + ik_y y\right),$$

and

$$p = p_o + p_1(z) \exp\left(i\sigma t + ik_x x + ik_y y\right).$$

From the linearized equation of motion show that

$$i\rho\left(\sigma + k_x U\right) w - B\left(ik_x b_z - \frac{db_x}{dz}\right) = -\frac{dp_1}{dz},$$

(we take units in which $\mu_o = 1$), and derive the corresponding equations for u and v.

From the linearized induction equation show that

$$b_x = \frac{k_x B}{\sigma + k_x U}\left\{u - \frac{iU'w}{\sigma + k_x U}\right\},$$

and obtain analogous expressions for b_y and b_z in terms of the perturbed velocity components.

Substitute these expressions for the perturbed components of the magnetic field into the linearized equations of motion.

From the $x-$ and $y-$components of the linearized equation of motion show that the z-component of the vorticity, $\zeta = ik_x v - ik_y u$, is given by

$$\zeta = \frac{k_y U' w}{\sigma + k_x U}.$$

Deduce that the y-component of the linearized equation of motion simplifies to

$$i\rho\left(\sigma + k_x U\right) v = -ik_y p_1.$$

Use div $\mathbf{u} = 0$ to show that

$$ik^2 p_1 = \rho\left(\sigma + k_x U\right)\frac{dw}{dz} - \rho k_x U' w,$$

where $k^2 = k_x^2 + k_y^2$.

Combine div $\mathbf{u} = 0$ with the expression for ζ to show that

$$ik^2 u = -\left[k_x \frac{dw}{dz} + \frac{k_y^2 U'}{\sigma + k_x U} w\right].$$

Hence, or otherwise, obtain an equation for w in the form

$$\frac{d}{dz}\left\{\rho\left(\sigma+k_xU\right)\frac{dw}{dz}-\rho k_xU'w\right\}$$

$$=k^2\rho\left(\sigma+k_xU\right)w+k_x^2B^2\left\{\frac{d}{dz}\left(\frac{dw/dz}{\sigma+k_xU}\right)-\frac{k^2w}{\sigma+k_xU}\right\}$$

$$-k_x^3B^2\frac{d}{dz}\left\{\frac{U'w}{\left(\sigma+k_xU\right)^2}\right\} \qquad (*)$$

Consider a shear layer at $z=0$, so that

$$U(z)=\begin{cases}U_2 & z>0\\ U_1 & z<0\,,\end{cases}$$

where U_1 and U_2 are constants. Show that in this case, the solutions of (*) for which $w/\left(\sigma+k_xU\right)$ is continuous and obey suitable boundary conditions as $|z|\to\infty$ are

$$w=\begin{cases}A\left(\sigma+k_xU_2\right)e^{-kz} & z>0\\ A\left(\sigma+k_xU_1\right)e^{kz} & z<0\end{cases}$$

for some constant A and for $k>0$.

By integrating (*) from $z=-\epsilon$ to $z=+\epsilon$ and letting $\epsilon\to 0$, show that

$$\rho\left(\sigma+k_xU_2\right)^2+\rho\left(\sigma+k_xU_1\right)^2=2k_x^2B^2.$$

Deduce that the shear flow is stable if $\left(U_1-U_2\right)^2<4B^2/\rho$.

[Hint: You may assume the identity: curl $(\mathbf{a}\wedge\mathbf{b})=(\mathbf{b}\,.\,\nabla)\mathbf{a}-(\mathbf{a}\,.\,\nabla)\mathbf{b}+\mathbf{a}$ div $\mathbf{b}-\mathbf{b}$ div $\mathbf{a}$.]