UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Wednesday 4 June 2003 1.30 to 4.30

PAPER 62

GALAXIES

Attempt **THREE** *questions.*

There are **four** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Show that the internal energy of a spherical star of radius R is

$$U = \frac{1}{\gamma - 1} \int_0^R P d\mathbf{r},$$

which can be written as

$$U = \frac{M_{tot}}{\gamma - 1} \int_0^1 \frac{P(m)}{\rho(m)} dm,$$

where

$$m = \frac{M(r)}{M_{tot}}$$

If we make adiabatic compressions or expansions of every spherical shell, then

$$P(m) = P_0(m) \left(\frac{\rho(m)}{\rho_0(m)} \right)^{\gamma},$$

where the subscript "0" refers to the initial state. For such perturbations, with the further simplifying assumption that each shell moves in a scaled fashion

$$\frac{r}{r_0} = \frac{R}{R_0},$$

show that

$$U = U_0 \left(\frac{R}{R_0} \right)^{-3(\gamma - 1)}.$$

Show also that under the same assumptions the gravitational energy scales as

$$W = W_0 \left(\frac{R}{R_0} \right)^{-1},$$

so that the total energy is

$$E = U + W = U_0 \left(\frac{R}{R_0} \right)^{-3(\gamma - 1)} + W_0 \left(\frac{R}{R_0} \right)^{-1}.$$

Draw $E(x)$ where $x = (R/R_0)$. Show that on the assumption that equilibrium $\left(\frac{\partial E}{\partial R} = 0 \right)$ corresponds to $R/R_0 = x = 1$, we must have, in equilibrium, that

$$3(\gamma - 1)U_0 = |W_0|$$

and check the condition that the equilibrium is stable,

$$\left(\frac{\partial^2 E}{\partial R^2} \right) > 0$$

implies

$$\gamma > 4/3\,.$$

2 Show that for a monatomic gas which is isothermal and self-gravitating in a planar distribution ($\rho = \rho(|z|)$) with a rms velocity, v_{rms}, that the density distribution satisfies the equation

$$v_z^2 \frac{d^2 \ln \rho}{dz^2} = -4\pi G \rho$$

which has the solution

$$\rho(z) = \rho_0 sech^2 \left(\frac{z}{2z_0} \right),$$

where

$$z_0 = \frac{\sigma_0}{\sqrt{8\pi a \rho_0}}; \qquad \text{and} \qquad \sigma_0^2 = \frac{1}{3} v_{rms}^2$$

Describe the nature of the general orbit in this potential.

3 Show that the gravitational potential energy of a spherical system can be written

$$W = -\frac{G}{2} \int_0^\infty \frac{M^2(r) dr}{r^2},$$

where $M(r)$ is the mass interior to radius r.

Derive the virial theorem for a self-gravitating system of N identical points, each of mass m. Define the gravitational radius R_g in terms of W by the formula:

$$W \equiv -G(mM)^2/R_g$$

Find the *rms* velocity dispersion $\langle v_{eq}^2 \rangle$ for a system in equilibrium [the answer to be phrased in terms of (n, m, R_g)].

Find the *rms* escape velocity $\langle v_{esc}^2 \rangle$ in the same units and also in terms of $\langle v_{eq}^2 \rangle$.

If a system is not in equilibrium, but collapses from being at rest at infinity and is observed in free fall at a time when its gravitational energy is W (defined above), what is its velocity dispersion $\langle v_{coll}^2 \rangle$ in units of $\langle v_{eq}^2 \rangle$?

4 Starting with the simple closed box model for the chemical evolution of a galaxy, and the solution

$$Z_*(\mu) = \frac{Y}{Y+1}\left[1 - \frac{\mu}{Y}\left(\frac{1-\mu^Y}{1-\mu}\right)\right]$$

where $Z_*(\mu)$ is the metallicity of a star born when the fraction of gas mass to total mass has been reduced from its initial value of unity to

$$\mu = \frac{M_g}{M_{g,init}}.$$

Show that, approximately (until most of the gas has been consumed, and the yield is small)

$$Z_*(\delta) = \frac{Y\delta}{2},$$

where δ is the fraction of the gas consumed: $\delta = 1-\mu$. Also show that if one examines the distribution of stars when the average star has the solar metallicity, $Z_\odot$, the most metal rich star has

$$Z_{*,max} = Z_\odot/2.$$

Find the distribution of stellar metallicities

$$dP = F(Z_*)dz_*.$$