UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2003 1.30 to 3.30

PAPER 61

BOUNDARY VALUE PROBLEMS FOR INTEGRABLE PDE'S

Attempt **TWO** *questions.*

There are **three** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Let $q(x,t)$ satisfy the following initial-boundary value problem

$$q_t + q_x - q_{xxx} = 0, \quad 0 < x < \infty, \quad t > 0, \tag{1}$$

$$q(x,0) = q_0(x), \quad 0 < x < \infty,$$

$$q(0,t) = g_0(t), \quad q_x(0,t) = g_1(t), \quad t > 0,$$

where $q_0(x)$ has sufficient decay as $x \to \infty$, $q_0(x)$, $g_0(t)$, $g_1(t)$, have sufficient smoothness, and $q_0(0) = g_0(0)$, $q_0'(0) = g_1(0)$.

(a) Write equation (1) in the form

$$\left(e^{-ikx+w(k)t}q\right)_t + \left(e^{-ikx+w(k)t}X\right)_x = 0, \quad k \in \mathbb{C}$$

where $w(k)$, $X(x,t,k)$ are to be determined.

(b) Find an integral representation for $q(x,t)$ in the complex k-plane involving appropriate spectral functions.

(c) Use the global relation to express the spectral functions in terms of the Fourier transform of $q_0(x)$ and of an appropriate t-transform of $g_0(t)$ and $g_1(t)$.

(d) Rewrite the spectral functions in a form suitable for analyzing the long time behaviour of the solution.

2 Let

$$M\varphi \doteqdot \varphi_{xx} - ik\varphi_x + q\varphi,$$

$$N\varphi \doteqdot \varphi_t + \varphi_{xxx} + 3q\varphi_x.$$

It can be shown that the KdV equation

$$q_t + q_{xxx} + 6qq_x = 0,$$

admits the Lax pair

$$M\varphi = 0, \quad N\varphi = 0.$$

Let $\varphi(k, x, t)$ be the *unique* solution of the linear integral equation

$$\varphi(k,x,t) + ie^{i(kx+k^3t)} \int_L \frac{\varphi(l,x,t)}{l+k} d\lambda(l) = e^{i(kx+k^3t)}, \tag{1}$$

where L and $d\lambda$ are appropriate contour and measure respectively. Define $q(x,t)$ by

$$q(x,t) = -\frac{\partial}{\partial x} \int_L \varphi(k,x,t) d\lambda(k). \tag{2}$$

(a) Show that $M\varphi$ satisfies the homogeneous version of (1).

(b) Show that $N\varphi$ satisfies an equation similar to (1) but with the r.h.s. $3ikM\varphi$ instead of $\exp[i(kx + k^3t)]$.

(c) Deduce that equations (1) and (2) provide a linearisation of the KdV equation.

(d) Use (1) and (2) to obtain a system of algebraic equations describing the n-soliton solution of the KdV.

(e) Write explicitly the 1-soliton solution.

3 It can be shown that the modified KdV equation

$$q_t - q_{xxx} + 6\lambda q^2 q_x = 0, \quad \lambda = \pm 1,$$

admits the following Lax pair

$$\mu_x - ik\hat{\sigma}_3\mu = Q\mu,$$
$$\mu_t + 4ik^3\hat{\sigma}_3\mu = \tilde{Q}\mu,$$

where

$$\hat{\sigma}_3\mu = [\sigma_3, \mu], \quad \sigma_3 = \text{ diag } (1, -1),$$
$$Q = \begin{pmatrix} 0 & q \\ \lambda q & 0 \end{pmatrix},$$
$$\tilde{Q} = \begin{pmatrix} -2i\lambda k q^2 & -4k^2 q + 2ikq_x - 2\lambda q^3 + q_{xx} \\ \lambda(-4k^2 q - 2ikq_x - 2\lambda q^3 + q_{xx}) & 2i\lambda k q^2 \end{pmatrix}.$$

Let q be a real valued function decaying as $x \to \infty$, which satisfies the modified KdV equation in $0 < x < \infty$, $0 < t < T$, where T is a positive constant.

(a) Let μ_1, μ_2, μ_3, be solutions of the above Lax pair normalized at $(0, T)$, $(0, 0)$, (∞, t) respectively. Find the domains in the complex k-plane where the column vectors of these matrices are bounded and analytic.

(b) Show that the functions μ_j, $j = 1, 2, 3$, are related by the equations

$$\mu_3(x, t, k) = \mu_2(x, t, k)e^{i(kx - 4k^3 t)\hat{\sigma}_3} s(k)$$

$$\mu_1(x, t, k) = \mu_2(x, t, k)e^{i(kx - 4k^3 t)\hat{\sigma}_3} S(k),$$

and express $s(k)$ and $S(k)$ in terms of $\mu_3(x, 0, k)$ and $\mu_2(0, t, k)$.

(c) Discuss how the equations obtained in (b) can be used to obtain a Riemann-Hilbert problem. What is the relevant contour for this Riemann-Hilbert problem?

(d) Show that there exists a simple relation between $s(k)$ and $S(k)$.

(e) By analyzing the linear limit of the relation obtained in (d), determine the number of the boundary conditions needed at $x = 0$ for the problem to be well posed (at least for a small norm of q).