UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Friday 30 May 2003 9 to 12

PAPER 60

SYMMETRIES AND PATTERNS

Attempt **THREE** *questions.*

There are **four** *questions in total.*

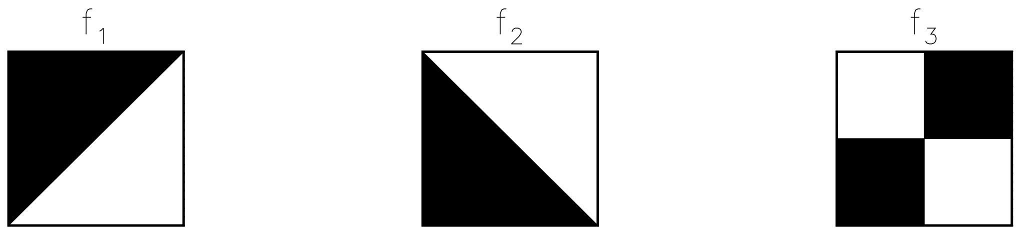
The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider steady-state bifurcation in a square box (i.e. with D_4 symmetry).

(i) Suppose first that the bifurcating solutions have the form $Af_1(x,y) + Bf_2(x,y)$, where the planforms f_1, f_2 break the symmetry about lines joining two opposite corners of the square, as in the diagram (black and white denote regions where the functions are positive and negative, respectively) and are related under the standard 2-dimensional action of D_4. Derive the normal form, and describe the possible steady solution branches in the neighbourhood of the bifurcation point in the non-degenerate case.

(ii) Now suppose that there is a degenerate situation in which there is another marginal solution of the form $Cf_3(x,y)$, where the planform f_3 is symmetric about the diagonals, but changes sign under rotation through $\pi/2$, as in the diagram.



By weakly breaking this degeneracy, derive equivariant evolution equations that have the form, correct to cubic order

$$
\begin{aligned}
A_T &= \mu_1 A - \alpha_0 A^3 - \lambda B^2 A - \alpha_1 AC - \alpha_2 AC^2 \\
B_T &= \mu_1 B - \alpha_0 B^3 - \lambda A^2 B + \alpha_1 BC - \alpha_2 BC^2 \\
C_T &= \mu_2 C - \gamma_0 C^3 + \gamma_1 (A^2 - B^2) - \gamma_2 C(A^2 + B^2)
\end{aligned}
$$

Now ignore the terms in α_2, γ_2, and assume that $\lambda > 1$, $\alpha_0 = \gamma_0 = 1$, $\alpha_1\gamma_1 > 0$. Investigate the stability of the branch with $A = B \neq 0$, $C = 0$ and show that this branch is unstable if μ_2 is too small, but is stable for a range of μ_2 if μ_1 is small enough. Identify the bifurcations at either end of this range, and hence show that it is generically possible to find oscillatory solutions of the system.

2 A supercritical pattern-forming instability in an infinite 1D layer is subject to a stationary imperfection proportional to $\cos k_c x$, where k_c is the critical wavenumber in the absence of the imperfection. Justify the use of the equation for the complex amplitude A

$$A_T = \mu A - |A|^2 A + \alpha,$$

where μ, α are real constants. Now suppose that instead of a stationary imperfection there is one that moves with a small speed c relative to the layer. By transforming into a frame moving at speed c, show that the appropriate equation is that given above but with the term μA replaced by $(\mu + i\Lambda)A$, where $\Lambda \propto c$ is a real constant.It may be supposed (without loss of generality) that α and Λ are positive, and that α may be set to unity by rescaling.

(i) Show (using the rescaled system $A_T = (\mu + i\Lambda)A - |A|^2 A + 1$) that values of $R \equiv |A|$ for steady states satisfy the cubic equation for R^2.

$$R^2(\mu - R^2)^2 + \Lambda^2 R^2 = 1.$$

Deduce, by considering the possible values of R^2 at the turning points (where $d\mu/dR = 0$), or otherwise that there can only be one such solution, whatever the value of μ, unless $\Lambda < \frac{\sqrt{3}}{2}$. Sketch bifurcation diagrams in the (μ, R) plane in the cases of large and small Λ.

(ii) Show that the eigenvalues $\sigma_{1,2}$ for perturbations to the steady state are given by the roots of the quadratic

$$\sigma^2 + 2\sigma(2R^2 - \mu) + (3R^2 - \mu)(R^2 - \mu) + \Lambda^2 = 0$$

Deduce that there are steady-state bifurcations only at the turning points (if these exist). Assuming that if there is no Hopf bifurcation for a particular value of Λ, there is none for any smaller value of Λ, show that there can be a Hopf bifurcation only if $\Lambda > 2^{-\frac{1}{3}}$. Indicate on bifurcation diagrams the number of unstable and stable eigenvalues at various points along the steady branch in the three cases *(a)* $\Lambda > \frac{\sqrt{3}}{2}$, *(b)* $\frac{\sqrt{3}}{2} > \Lambda > 2^{-\frac{1}{3}}$, *(c)* $\Lambda < 2^{-\frac{1}{3}}$.

3 A pattern-forming instability in the presence of a conserved quantity can be shown to obey the coupled equations for the real variables $A(X,T), B(X,T)$ (supposed periodic in $-L < X < L$)

$$A_T = \mu A + \alpha A^2 - A^3 + A_{XX} - AB$$
$$B_T = \sigma(B + \delta A^2)_{XX}$$

Here A represents the envelope of the pattern, and B the conserved quantity. B is normalised by the requirement that $\langle B \rangle \equiv (2L)^{-1} \int_{-L}^{L} B\, dX = 0$. It may be assumed that δ and σ are positive.

(i) Defining the periodic function H by $B = dH/dX$, show that these equations are variational, in that the quantity

$$\mathcal{V} \equiv \left\langle \frac{\mu}{2}A^2 + \frac{\alpha}{3}A^3 - \frac{1}{4}A^4 - \frac{1}{2}A_X^2 - \frac{1}{2}A^2 H_X - \frac{1}{4\delta}H_X^2 \right\rangle$$

satisfies

$$\frac{d\mathcal{V}}{dT} = \left\langle A_T^2 + \frac{1}{2\delta\sigma}H_T^2 \right\rangle .$$

By using the Schwartz inequality, show that

$$\left\langle -\frac{1}{2}A^2 H_X - \frac{1}{4\delta}H_X^2 \right\rangle \leqslant \frac{1}{2}\left\langle A^4 \right\rangle^{\frac{1}{2}} \left\langle H_X^2 \right\rangle^{\frac{1}{2}} - \left\langle \frac{1}{4\delta}H_X^2 \right\rangle \leqslant \frac{\delta}{4}\left\langle A^4 \right\rangle ,$$

and hence show that $\mathcal{V}$ is bounded above when $0 < \delta < 1$. What do you deduce about solutions at long times?

(ii) Find conditions on μ, δ for the stability of the steady uniform solution branch given by $0 = \mu + \alpha A - A^2 = B$. Show in particular that for any δ and $\alpha \neq 0$ there is a range of μ in which the solutions are unstable even when $dA/d\mu = 0$, and that all such solutions are unstable if $\delta > 1$.

(iii) Now suppose $L \gg 1$. Steady non-uniform solutions of the equations are given by

$$0 = (\mu - \delta\langle A^2 \rangle)A + \alpha A^2 - (1-\delta)A^3 + A_{XX} \qquad (*)$$
$$B = \delta(\langle A^2 \rangle - A^2)$$

A localised solution of the steady equations can be approximated for large L by $A \approx R$, $-hL < X < hL (0 < h < 1)$, $A \approx 0$ otherwise. Show that $\langle A^2 \rangle \approx hR^2$ as long as h is not too close to 0 or 1. By integrating equation $(*)$, show that the condition $2\alpha^2 = -9(\mu - \delta\langle A^2 \rangle)(1-\delta)$ must hold and hence derive an equation for h as a function of μ, and give an approximate expression for the range of existence of localised solutions for $0 < \delta < 1$.

4 Write an essay on the Landau-Ginzburg equation. You should include derivation via perturbation theory and symmetry arguments, steady solution types, stability of constant amplitude solutions and extensions to another space dimension (with critique).