UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2003 9 to 12

# PAPER 6

# INTRODUCTION TO BANACH SPACES AND ALGEBRAS

*Attempt* **THREE** *questions.*

*There are* **five** *questions in total.*

*The questions carry equal weight.*

**You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.**

**1** Let $E$ be a Banach space, $F$ a normed space and let $T$ be a bounded linear mapping from $E$ to $F$. Suppose that there exist constants $R > 0$ and $0 < k < 1$ such that, for every $y \in F$ with $\|y\| \leqslant 1$, there is some $x \in E$ with $\|x\| \leqslant R$ such that $\|Tx - y\| \leqslant k$. Prove that (i) for every $y \in F$ there is some $x \in E$ with $Tx = y$ and $\|x\| \leqslant \left(\frac{R}{1-k}\right)\|y\|$ and (ii) $F$ is complete.

State, *without proof*, the Open Mapping Theorem, for a continuous linear map between two Banach spaces. Deduce the Closed Graph Theorem.

Let $E, F$ be Banach spaces and let $T : E \to F$ be a linear mapping. Define a subset $S$ of $F$ by saying that an element $y$ of $F$ belongs to $S$ if and only if there is some sequence $(x_n)$ in $E$ such that $x_n \to 0$ in $E$, while $Tx_n \to y$ in $F$. Prove that $S$ is a closed linear subspace of $F$, and that $T$ is continuous if and only if $S = \{0\}$.

**2** Let $E$ be a real vector space and let $K$ be a convex subset of $E$. Define what it means to say that a point $x \in K$ is an *extreme point* of $K$.

Now let $E$ be a real, Hausdorff, locally convex topological vector space and let $K$ be a compact, convex subset of $E$. Prove that $K$ is the closed convex hull of its set of extreme points. [*You may assume, without proof, any version of the Hahn-Banach theorem, and of the separation theorem for convex sets*.]

Define the standard real Banach spaces $\ell^1$ and $\ell^\infty$, and identify the extreme points of their closed unit balls. [*You should assume, without proof, that these spaces* are *Banach spaces*.]

**3** Let $A$ be a Banach algebra, with identity, over the complex field, and let $x \in A$. Define the *spectrum*, $\operatorname{Sp} x$, of $x$ in $A$. Prove that $\operatorname{Sp} x$ is a non-empty, compact subset of $\mathbb{C}$. [*You should assume, without proof, that*: (i) *the set $G(A)$ of invertible elements of $A$ is open in $A$, and* (ii) *the mapping $x \mapsto x^{-1}$ ($x \in G(A)$) is continuous*.]

Let $E = \mathcal{O}(\mathbb{C})$ be the algebra of all complex-valued, holomorphic (i.e. analytic) functions on the complex plane. Show that $E$ can be given an algebra-norm, but not a complete algebra-norm.

Let $D = C(\mathbb{C})$ be the algebra of all complex-valued, continuous functions on the complex plane. Explain, briefly, how to find, in $D$, a function $f$ and a sequence $(g_n)$ of non-identically-zero functions, such that $fg_n = ng_n$, for all $n \geqslant 1$. Deduce that $D$ cannot be given any algebra-norm.

**4** Let $A$ be a commutative Banach algebra, with identity, over the complex field. Let $x \in A$, let $U$ be an open neighbourhood of $\operatorname{Sp} x$ in $\mathbb{C}$ and let $\mathcal{O}(U)$ be the algebra of all complex-valued, holomorphic functions on $U$, in its standard topology (i.e. local uniform convergence). Prove that there is a unique, continuous, unital homomorphism $\Theta_x : \mathcal{O}(U) \to A$ such that $\Theta_x(Z) = x$ (where $Z$ is the coordinate function $Z(\lambda) = \lambda$ on $U$). [*You may assume, without proof, any form of the Runge approximation theorem.*]

Prove also that, for every character $\phi$ on $A$ and every $f \in \mathcal{O}(U)$, we have $\phi\big(\Theta_x(f)\big) = f\big(\phi(x)\big)$.

An element $e$ of $A$ is called *idempotent* if and only if $e^2 = e$. Suppose that $A$ contains an element $a$ such that $\operatorname{Sp} a = X \cup Y$, where $X, Y$ are disjoint, non-empty, closed subsets of $\mathbb{C}$. Prove that $A$ contains an idempotent $e$ such that, for every character $\phi$, $\phi(e) = 1$ whenever $\phi(a) \in X$ and $\phi(e) = 0$ whenever $\phi(a) \in Y$.

**5** Give an account of the elementary theory of $C^*$-algebras, leading to a proof of the Gelfand-Naimark theorem for commutative $C^*$-algebras.

Let $x$ be a normal element of a $C^*$-algebra $A$. Prove that there is a unique continuous, unital $*$-homomorphism $\theta_x : C(\operatorname{Sp} x) \to A$ such that $\theta_x(Z) = x$ (where $Z$ is the coordinate function $Z(\lambda) = \lambda$ on $\operatorname{Sp} x$). Show, also, that $\theta_x$ is isometric, and that the image of $\theta_x$ is the closed subalgebra of $A$ generated by $\{x, x^*\}$.

Prove that there is a unique hermitian element $h$ of $A$ such that both $h^2 = x^*x$ and $\operatorname{Sp} h \subset \mathbb{R}^+$.