

PAPER 59

LOCAL AND GLOBAL BIFURCATIONS

*Attempt **THREE** questions.*

*There are **four** questions in total.*

*The questions carry equal weight.*

**You may not start to read the questions  
printed on the subsequent pages until  
instructed to do so by the Invigilator.**

1 The following set of ordinary differential equations (ODEs) defines a dynamical system on a two-dimensional spherical surface:

$$\begin{aligned}\dot{\theta} &= \sin \theta [\cos \theta (-\lambda + \cos 2\phi) - \sin 2\phi], \\ \dot{\phi} &= \cos \theta (\kappa - \cos 2\phi) - \sin 2\phi,\end{aligned}\tag{1}$$

where  $\theta$  and  $\phi$  are the usual spherical polar co-ordinates denoting latitude and longitude and  $\lambda$  and  $\kappa$  are real parameters. Through periodicity of the dynamics we may restrict our attention to the half of the upper hemisphere given by  $0 < \theta \leq \frac{\pi}{2}$  and  $0 \leq \phi < \pi$ .

(a) Show that the system (1) has equilibria at the points  $P_1 : (\theta, \phi) = (\frac{\pi}{2}, 0)$  and  $P_2 : (\theta, \phi) = (\frac{\pi}{2}, \frac{\pi}{2})$ .

(b) Investigate local bifurcations from  $P_1$  and show that a codimension-two bifurcation occurs at the point  $(\lambda, \kappa) = (3, -1)$ . Sketch the bifurcation curves in the  $(\lambda, \kappa)$  plane.

(c) By suitable expansions, a linear co-ordinate change, and a time rescaling, show that the dynamics of (1) at the codimension-two point can be written in the form

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u, v) \\ g(u, v) \end{pmatrix} + O(4),\tag{2}$$

where  $f(u, v)$  and  $g(u, v)$  contain only cubic terms and  $O(4)$  denotes terms of order 4 and higher jointly in  $u$  and  $v$ .

(d) You may assume that, after a suitable near-identity transformation and a rescaling have been applied to (2), it can be put in the form

$$\begin{aligned}\dot{x} &= y, \\ \dot{y} &= \mu_1 y - \mu_2 x + x^3 - x^2 y.\end{aligned}\tag{3}$$

Briefly discuss the local and global bifurcations of (3). You may assume that the Hopf bifurcation is supercritical. Hence justify the existence of exactly one curve of global bifurcations for the original ODEs (1) near  $(\lambda, \kappa) = (3, -1)$ .

(e) Sketch the phase portrait of (3) in each of the four regions of the  $(\mu_1, \mu_2)$  plane that contains qualitatively distinct behaviour.

**2** Consider two-dimensional Boussinesq thermal convection in a region  $0 \leq x \leq L$ ,  $0 \leq z \leq 1$ , with an imposed uniform rotation  $\Omega$  about a vertical ( $z$ ) axis. With the assumption that the velocity  $\mathbf{u} = (u, v, w)$  and temperature  $T$  depend only on  $x$ ,  $z$  and time  $t$ , the dimensionless PDEs that describe the behaviour of the fluid are:

$$\begin{aligned}\frac{\partial}{\partial t} \nabla^2 \psi + \frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, z)} &= \sigma \nabla^4 \psi + \sigma R \frac{\partial \theta}{\partial x} - \sigma R_\Omega \frac{\partial v}{\partial z}, \\ \frac{\partial \theta}{\partial t} + \frac{\partial(\psi, \theta)}{\partial(x, z)} &= \nabla^2 \theta + \frac{\partial \psi}{\partial x}, \\ \frac{\partial v}{\partial t} + \frac{\partial(\psi, v)}{\partial(x, z)} &= \sigma \nabla^2 v + \sigma R_\Omega \frac{\partial \psi}{\partial z},\end{aligned}$$

where  $\psi(x, z, t)$  is the streamfunction for the  $x$  and  $z$  components of the velocity field and the temperature  $T = 1 - z + \theta(x, z, t)$ .  $\sigma$  is the Prandtl number,  $R$  is the Rayleigh number (proportional to the imposed temperature difference) and  $R_\Omega$  is the Taylor number (proportional to the rotation rate  $\Omega$  which determines the Coriolis force on the fluid). The boundary conditions imposed are  $\psi = \nabla^2 \psi = 0$  on all four walls,  $\theta = \frac{\partial v}{\partial z} = 0$  on  $z = 0, 1$  and  $\frac{\partial \theta}{\partial x} = v = 0$  on  $x = 0, L$ .

(a) Adopt the truncated representation

$$\begin{aligned}\psi &= \frac{2\sqrt{2}\beta}{\alpha} a(t) \sin \alpha x \sin \pi z, \\ \theta &= \frac{2\sqrt{2}}{\beta} b(t) \cos \alpha x \sin \pi z - \frac{1}{\pi} c(t) \sin 2\pi z, \\ v &= \frac{2\sqrt{2}\sigma\pi R_\Omega}{\alpha\beta} d(t) \sin \alpha x \cos \pi z + \frac{\sigma R_\Omega}{\alpha} e(t) \sin 2\alpha x,\end{aligned}$$

where  $\alpha = \pi/L$  and  $\beta^2 = \alpha^2 + \pi^2$ , and derive the truncated model ODEs for weakly nonlinear rotating convection:

$$\begin{aligned}\dot{a} &= -\sigma a + \sigma r b - \sigma^2 r_\Omega^2 d, \\ \dot{b} &= a - b - ac, \\ \dot{c} &= \varpi(-c + ab), \\ \dot{d} &= a - \sigma d - ae, \\ \dot{e} &= -(4 - \varpi)\sigma e + \varpi ad,\end{aligned}$$

where the dot represents  $d/d\tau$ ,  $\tau = \beta^2 t$ ,  $\varpi = 4\pi^2/\beta^2$  and  $r$  and  $r_\Omega$  are proportional to  $R$  and  $R_\Omega$ . Determine the constants of proportionality.

(b) Locate the codimension-one and Takens-Bogdanov bifurcations from the trivial solution, assuming that  $\sigma < 1$ . Sketch the bifurcation curves in the  $(r, r_\Omega^2)$  plane.

(c) Determine the value of  $r_\Omega^2$  at which the pitchfork bifurcation changes from being supercritical to being subcritical. Discuss the bifurcation structure in the vicinity of this point.

(d) Investigate the possibility of a codimension-three bifurcation.

**3** Consider the system of ODEs:

$$\begin{aligned} \dot{x} &= x + P(x, y; \mu, \nu), \\ \dot{y} &= -\delta y + Q(x, y; \mu, \nu), \end{aligned} \tag{1}$$

where  $\delta > 1$  is a constant,  $\mu$  and  $\nu$  are parameters, and the functions  $P$  and  $Q$  are purely nonlinear in  $x$  and  $y$ , i.e.

$$\begin{aligned} P(0, 0; \mu, \nu) &= \frac{\partial P}{\partial x}(0, 0; \mu, \nu) = \frac{\partial P}{\partial y}(0, 0; \mu, \nu) = 0, \\ Q(0, 0; \mu, \nu) &= \frac{\partial Q}{\partial x}(0, 0; \mu, \nu) = \frac{\partial Q}{\partial y}(0, 0; \mu, \nu) = 0. \end{aligned}$$

Assume that when  $\mu = 0$  there is a homoclinic orbit that leaves the origin with  $x > 0$  and returns to the origin with  $y > 0$ , while for  $\nu = 0$  there is a homoclinic orbit that leaves the origin with  $x < 0$  and returns to the origin with  $y < 0$ .

(a) Show that the dynamics of the ODEs (1) near the origin can be described by iterations of the 2D map

$$\begin{aligned} x_{n+1} &= \begin{cases} -\mu + A \operatorname{sgn}(y_n) |x_n|^\delta & x_n > 0 \\ \nu + B \operatorname{sgn}(y_n) |x_n|^\delta & x_n < 0 \end{cases} \\ y_{n+1} &= \operatorname{sgn}(x_n) \end{aligned}$$

assuming that  $\mu$  and  $\nu$  are both small,  $A$  and  $B$  are constants, and we define the function  $\operatorname{sgn}(z) = z/|z|$  for  $z \neq 0$ ,  $\operatorname{sgn}(0) = 0$ .

(b) Indicate the regions in the  $(\mu, \nu)$  plane in which there are one or two periodic orbits, paying particular attention to the locations of the global bifurcations. Sketch a phase portrait of (1) in each region of the  $(\mu, \nu)$  plane that contains qualitatively distinct behaviour, and also at a point on each global bifurcation curve.

(c) Discuss how the results of the analysis of parts (a) and (b) relate to the analysis of the planar gluing bifurcation.

4 The dynamics near a particular codimension-two bifurcation point are given by the ODEs:

$$\begin{aligned}\dot{z} &= (\mu_1 + i)z - 2xz, \\ \dot{x} &= \mu_2 + x^2 + |z|^2,\end{aligned}\tag{1}$$

where  $z \in \mathbb{C}$  and  $x \in \mathbb{R}$ .  $\mu_1$  and  $\mu_2$  are real bifurcation parameters.

(a) By transforming to polar co-ordinates  $z = re^{i\theta}$ , describe fully the local bifurcations from the equilibria of (1), and sketch a bifurcation diagram in the  $(\mu_1, \mu_2)$  plane.

(b) Identify a secondary bifurcation in (1) and show that it is a degenerate bifurcation. You may find the function  $F(r, x, \mu_2) = -r(\mu_2 + \frac{1}{3}r^2 + x^2)$  useful.

(c) This degeneracy is broken by the addition of the term  $x^2z$  to the  $\dot{z}$  equation in (1). Including this term, use the rescaling  $r = \varepsilon u$ ,  $x = \varepsilon v$ ,  $\mu_1 = \varepsilon^2 \lambda_1$ ,  $\mu_2 = \varepsilon^2 \lambda_2$ ,  $\tau = \varepsilon t$  to deduce ODEs for  $u$  and  $v$  which have the conserved quantity  $F(u, v, \lambda_2)$  in the limit  $\varepsilon \rightarrow 0$ . Sketch contours of constant  $F$  in the  $(u, v > 0)$  half-plane, in the case  $\lambda_2 < 0$ . Give the value of  $F$  that corresponds to the heteroclinic orbits.

(d) By integrating around the heteroclinic orbit for small  $\varepsilon$ , find the relationship between  $\lambda_1$  and  $\lambda_2$ , and hence between  $\mu_1$  and  $\mu_2$ , at the global bifurcation. You may find the substitution  $v = \sqrt{-\lambda_2} \cos \phi$  useful. Indicate this global bifurcation curve on your sketch from part (a).

(e) Discuss the generation of complex dynamics near the global bifurcation when generic higher-order terms are added to (1). Your discussion should include the role of the normal form symmetry, and the dynamics of the Shilnikov map:

$$y_{n+1} = -\mu + Ay_n^\delta \cos(B \log(y_n) + \Phi),$$

in the case  $\delta < 1$ , where  $A$ ,  $B$  and  $\Phi$  are constants and  $\mu$  is the bifurcation parameter.