UNIVERSITY OF CAMBRIDGE

MATHEMATICAL TRIPOS Part III

Monday 2 June 2003 1.30 to 4.30

PAPER 58

ADVANCED COSMOLOGY

Attempt **THREE** *questions.*

There are **six** *questions in total.*

The questions carry equal weight.

You may not start to read the questions printed on the subsequent pages until instructed to do so by the Invigilator.

1 Consider a FRW Universe with scale factor a and spatial curvature k/a^2, containing matter with density ρ_m, radiation with density ρ_r, and a cosmological constant with density ρ_Λ.

(i) If $\rho_\Lambda < 0$, show by sketching the effective potential for a that an expanding universe always recollapses. Show that its lifetime is bounded above by $\sqrt{\frac{3\pi}{8G|\rho_\Lambda|}}$ where G is Newton's constant.

(ii) If $\rho_\Lambda > 0$, show that the universe may evolve in five qualitatively different ways, one of which is to remain static and two of which are related by time reversal.

(iii) Show that in the static case, one must have

$$\rho_m + 2\rho_r = 2\rho_\Lambda\,.$$

(iv) If $k = \rho_r = 0$, and $\rho_{m,0}$ is the matter density today, at what redshift did the expansion of the universe start to accelerate?

2 Under an infinitesimal coordinate (gauge) transformation, $x'^{\mu} = x^{\mu} + \xi^{\mu}(x), \mu = 0, 1, 2, 3$, the spacetime metric transforms as

$$g'_{\mu\nu}(x) - g_{\mu\nu}(x) = -g_{\mu\alpha}\xi^{\alpha}_{,\nu} - g_{\nu\alpha}\xi^{\alpha}_{,\mu} - g_{\mu\nu,\alpha}\xi^{\alpha}\,.$$

(i) Show that if one has $g_{\mu\nu} = a^2(\tau)(\eta_{\mu\nu} + h_{\mu\nu}(x))$ with $\eta_{\mu\nu} = \text{diag}\,(-1, 1, 1, 1)$ and $|h_{\mu\nu}| \ll 1$, i.e. a perturbed flat FRW universe, where $x^0 = \tau$ is the conformal time, then under a coordinate transformation of the form

$$\xi^{\mu} = (T(\tau), i\hat{k}_i L(\tau))e^{i\mathbf{k}\cdot\mathbf{x}}$$

(where $i = 1, 2, 3$ and $\hat{k}_i = k_i/|\mathbf{k}|$), $h_{\mu\nu}$ changes as follows

$$\begin{aligned} h'_{00} &= h_{00} + 2(\dot{T} + \frac{\dot{a}}{a}T) \\ h'_{0i} &= h_{0i} + i\hat{k}_i(kT - \dot{L}) \\ h'_{ij} &= h_{ij} + 2(Lk\hat{k}_i\hat{k}_j - \frac{\dot{a}}{a}T\delta_{ij})\,, \end{aligned} \tag{1}$$

where dots denote derivatives with respect to τ.

(ii) Synchronous gauge is defined by $h_{00} = h_{0i} = 0$, $i = 1, 2, 3$. For scalar perturbations of wave vector $\mathbf{k}$ we have

$$h_{ij}(\tau) = \left[\frac{1}{3}h(\tau)\delta_{ij} + h_s(\tau)(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij})\right]e^{i\mathbf{k}\cdot\mathbf{x}}\,.$$

Conformal Newtonian gauge is defined by $h_{0i} = 0, h_{ij} = -2\Psi\delta_{ij}, h_{00} = -2\Phi$.

Find the gauge transformation $\xi^{\mu}(x)$ which takes you from synchronous to Newtonian gauge. Hence show that

$$\Phi = \frac{\ddot{h}s + \frac{\dot{a}}{a}\dot{h}s}{2k^2} \qquad \Psi = -\frac{1}{6}(h - hs) - \frac{\dot{a}}{a}\frac{\dot{h}s}{2k^2} \tag{2}$$

where dots denote τ derivatives.

(iii) Show that synchronous gauge is preserved under transformations (1) with $T = \frac{1}{a}f$ and $L = k\int \frac{d\tau}{a(\tau)}f + g$ with f and g arbitrary constants for each $\mathbf{k}$.

Find the transformation rule for h and h_s and hence show that Φ and Ψ given in (2) are invariant under these transformations.

3 (a) The trace of the Einstein equation in synchronous gauge is given by

$$\ddot{h} + \frac{\dot{a}}{a}\dot{h} + 3\left(\frac{\dot{a}}{a}\right)^2 \sum_N (1 + 3w_N)\delta_N \Omega_N = 0\,,$$

where h is the trace of the metric perturbation, a is the scale factor, w_N is the equation of state of each component fluid and the derivatives are with respect to the comoving time, τ. Use this and the conservation of stress energy to find the solutions of $\delta_m(\tau)$ in the matter dominated regime.

(b) Use this to derive the power spectrum of scale invariant fluctuations, where scale invariance implies that the contribution to the variance $< \delta^2 >$ per log k interval is constant on horizon entry ($k\tau \simeq 2\pi$).

(c) Consider a universe with matter and radiation. What is the equation for the evolution of the growth of matter perturbations? Change from comoving time to $a/a_{\rm eq} \equiv \eta$ as the independent variable. Show that a solution is $\delta_m = A(\vec{x})(1 + 3\eta/2)$, assuming that the oscillating radiation perturbations can be ignored, i.e. $< \delta_\gamma >= 0$ on sub-horizon scales. What does this imply for the shape of the present matter power spectrum?

4 The Sachs–Wolfe equation is given by

$$\frac{\delta T}{T}(\hat{n}) = \frac{1}{4}\delta_\gamma - \vec{v}\cdot\hat{n} - \frac{1}{2}\int d\tau \dot{h}_{ij} n_i n_j\,,$$

where the sum over Fourier modes is implicit on the right hand side.

(a) Explain the origin of each of the terms on the right hand side.

(b) Show that on very large scales $\frac{\delta T}{T}(\hat{n}) \simeq \frac{1}{2k^2}\ddot{h}_s$.

(c) Using Poisson's equation, show this implies $\frac{\delta T}{T}(\hat{n}) \simeq \frac{1}{3}\Phi$ for the growing mode $\delta_m = Ak^{\frac{n}{2}}\tau^2$.

(Assume $\dot{h} = \dot{h}_s$, i.e. no anistropic stress.)

(d) Use this to calculate $a_{\ell m} = \int d\Omega \frac{\delta T}{T}(\hat{n}) Y^*_{\ell m}(\hat{n})$.

(e) Finally find the dependence of $C_\ell =< |a_{\ell m}|^2 >$ as a function of ℓ for a scale invariant ($n = 1$) spectrum.

Useful relations

$$e^{i\vec{k}\cdot\vec{r}} = 4\pi \sum_{\ell m} i^\ell j_\ell(kr) Y_{\ell m}(\hat{r}) Y^*_{\ell m}(\hat{k})\,,$$

$$\int_0^\infty k^{-1} dk j_\ell(k\tau) = \frac{1}{2\ell(\ell+1)}\,.$$

5 (a) Consider the metric

$$ds^2 = -dt^2 + \cosh^2[t]\, d\xi^2 \,,$$

where $-\infty < t < +\infty$ and $-\infty < \xi < +\infty$, are coordinates covering all of $(1+1)$-dimensional de Sitter space. Find the transformation to null coordinates u, v, so that the metric takes the form

$$ds^2 = f^2(u,v)\; du\; dv$$

for some function $f(u,v)$.

Hint: You may use the fact that

$$\int \frac{d\zeta}{\cosh[\zeta]} = 2\tan^{-1}[\tanh(\zeta/2)].$$

(b) Use the result from (a) to determine the causal structure of the spacetime. Draw a Penrose (conformal) diagram for this spacetime. Is the spacetime globally hyperbolic? Explain your answer and its physical implications for the initial value problem.

(c) For a classical massless scalar field ϕ in $(3+1)$-dimensional de Sitter space expressed in terms of the flat coordinates

$$ds^2 = -dt^2 + \exp[2t]\left(dx_1^2 + dx_2^2 + dx_3^2 \right),$$

described by the action

$$S = \int d^4x \sqrt{-g}\; g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi)\,,$$

find the normal mode solutions obtained using separation of variables.

Hint: You may find it convenient to transform to conformal time. The spherical Bessel functions $j_\ell(z)$ and $y_\ell(z)$ satisfy the equation

$$\frac{d^2F}{dz^2} + \frac{2}{z}\frac{dF}{dz} + \left(1 - \frac{\ell(\ell+1)}{z^2}\right) F = 0.$$

6 Consider electrostatics in a five-dimensional Kaluza-Klein geometry constructed as the direct product of ordinary (3+1)-dimensional Minkowski space and a circle of radius R for the extra curled-up dimension, so that the metric takes the form

$$ds^2 = -dt^2 + dx_1{}^2 + dx_2{}^2 + dx_3{}^2 + dz^2$$

where $z = 0$ and $z = 2\pi R$ have been identified. Suppose that the Poisson equation for the electrostatic potential takes the form

$$\nabla^2 \phi = \; -4\pi R \rho.$$

Here ∇^2 is the four-dimensional Laplacian.

What is the potential energy as a function of the separation r between two charges of magnitude q_1 and q_2 both located at the same value of z but separated by a distance r in the three infinite spatial dimensions? What is the limit of your result for $r \gg R$ and for $r \ll R$? Explain how if the electromagnetic field were free to explore the finite fifth dimension, as we have assumed in this problem, one might construct an experiment to use your result to uncover the presence of such a fifth dimension.